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MHD stability analysis via matching method including finite Larmor radius effect

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Special Thanks to
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Outline

I. Background and Motivation
II. Ordering scheme in outer region including FLR effects
III. Numerical results
IV. Conclusions

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I. Background and Motivation

Background: Matching method for boundary-layer problems

- Traditionally, matched asymptotic expansion has been used

  [H. P. Furth, J. Killeen and M. N. Rosenbluth, Phys. Fluids **6**, 459 (1963).]

  [B. Coppi, J. M. Greene, and J. L. Johnson, Nucl. Fusion **6**, 101 (1966).]

- The matched asymptotic expansion has some difficulties:
  - Reversed magnetic shear plasmas: Irregular singularity
    - Failure in Frobenius series solution
  - Plasmas marginally stable against ideal MHD: Remaining $E_{||}$ in outer region
    - Less accurate approximation by inertialess, ideal MHD (Newcomb eq.) in outer region
  - Compatibility with numerical computations
    - Error in calculating matching data in outer region
    - Error in handling unbounded domain in inner layer

- Some sophisticated theories have been developed

  [A. Pletzer and R. L. Dewar, J. Plasma Phys. **45**, 427 (1991).]

  [R. L. Dewar and M. Persson, Phys. Fluids B **5**, 4273 (1993).]

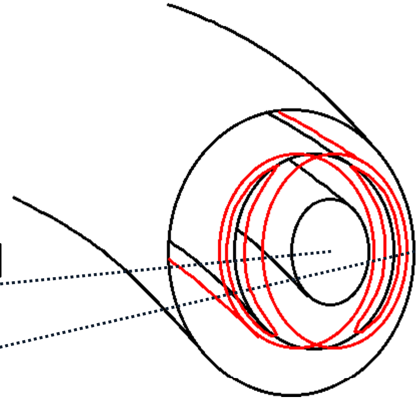
  [A. Pletzer, A. Bondeson and R. L. Dewar, J. Comput. Phys. **115**, 530 (1994).]

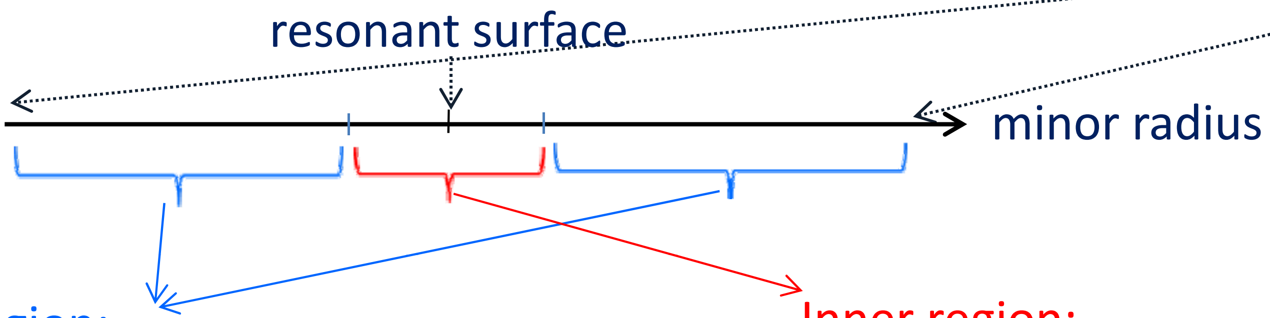
  [S. Tokuda and T. Watanabe, Phys. Plasmas **6**, 3012 (1999).]

  [S. Tokuda, Nucl. Fusion **41**, 1037 (2001).]

Background: Matching via inner region with finite width

- We have developed a new matching method, which utilizes an inner region with a finite width
  - Firstly developed for m/n=1/1 ideal internal kink mode,
    [Y. Kagei and S. Tokuda, Plasma Fusion Res. **3**, 039 (2008).]
  - and extended to resistive modes
    [M. Furukawa, S. Tokuda and L. –J. Zheng, Phys. Plasmas **17**, 052502 (2010).]





Outer region:
We solve the linearized MHD equations without plasma inertia and resistivity

Inner region:
We solve the linearized MHD equations including plasma inertia and resistivity

Conditions across the matching boundaries

(1) Continuity of perturbed magnetic field

(2) Smooth disappearance of $E_{||}$ as going away from the inner region to the outer region

This was a non-trivial extension, stemming from difference of order of spatial differentiation between outer and inner regions due to resistivity

Significance of finite-width inner region

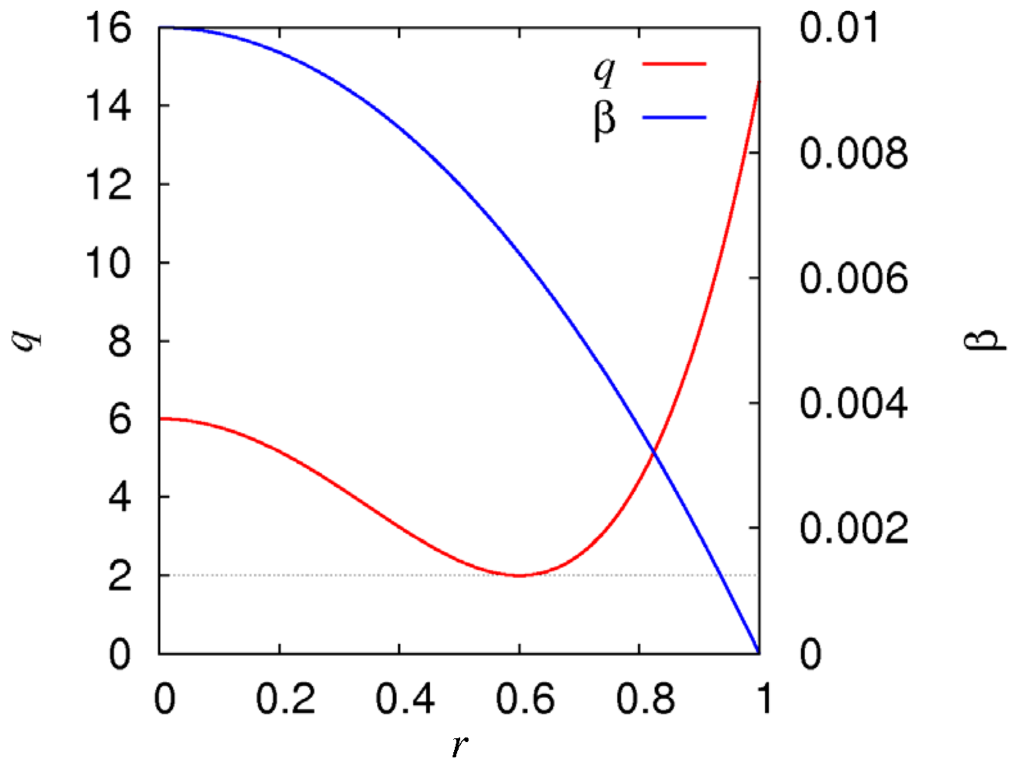
- Since the matching is NOT asymptotic,
  - No need to rely on Frobenius series solutions around the singularity
    - Applicable to irregular singularity cases
  - Plasmas marginally stable against ideal MHD
    - will be resolved in a few slides later
  - Compatible with numerical computations (no need to take LIMIT numerically)
    - We do not need to calculate matching data
    - We do not need to handle unbounded domain

- **Therefore, the new matching method using finite-width inner region enables us to AVOID, not treat better, difficulties of the traditional method**

Example: m/n=2/1 (double tearing) mode

- Aspect ratio $A = 10$
- 800 grids are used in the whole domain

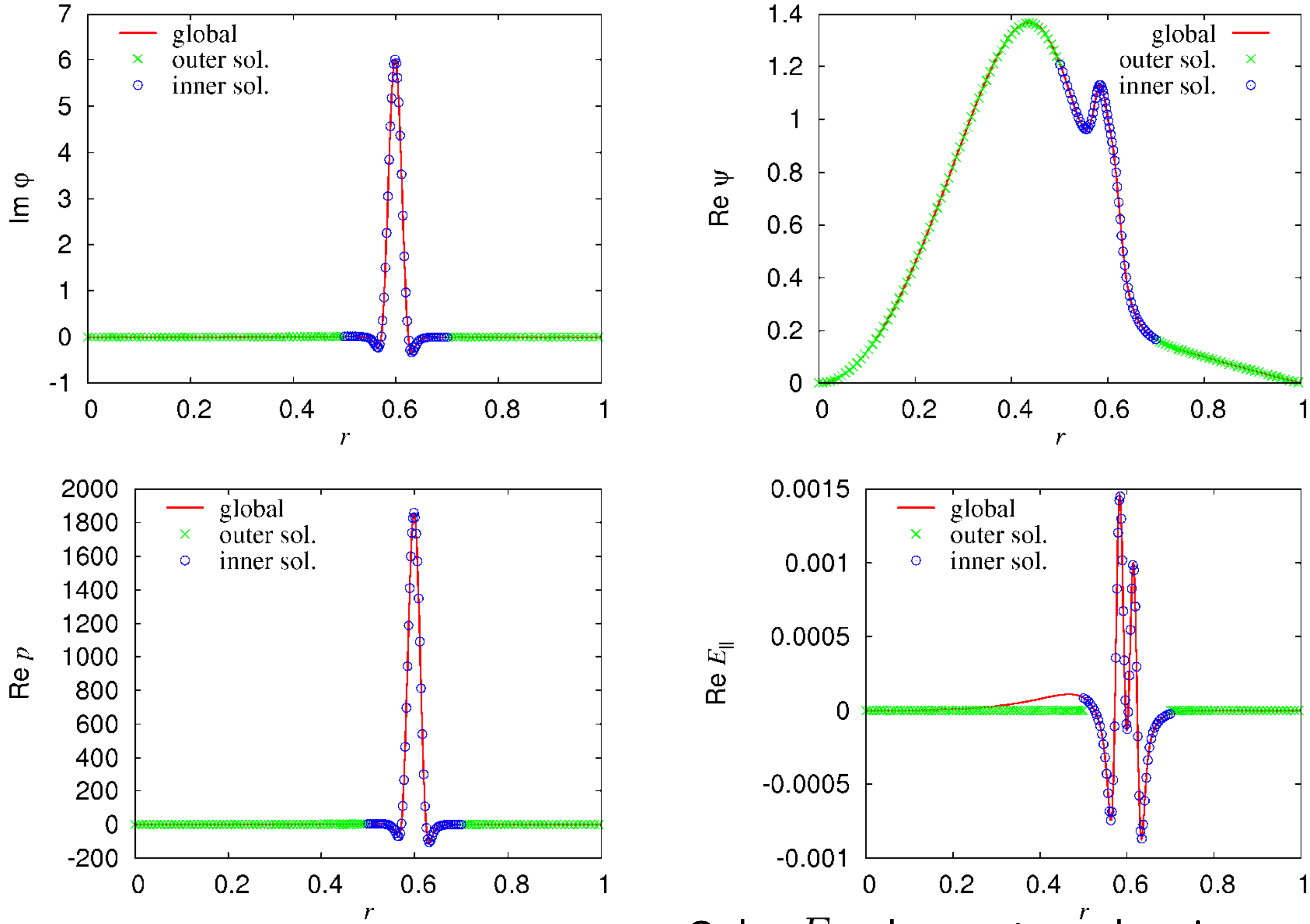
(typical q and β profiles)



Example: Eigenfunction of m/n=2/1 mode

$q_{\min} = 2$

$\eta = 10^{-6}$

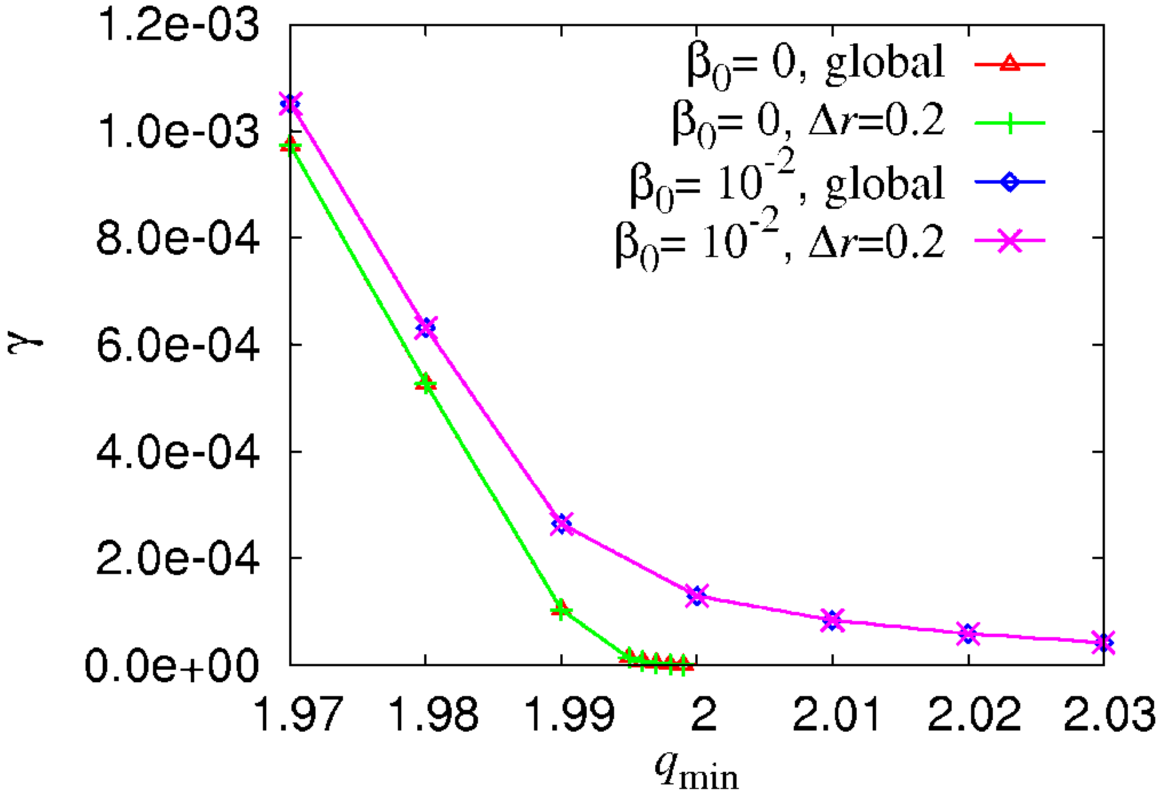


Almost complete overlap is observed

Only $E_{\|}$ does not overlap since we solved ideal MHD equation in the outer region

Example: Growth rate of m/n=2/1 (double tearing) mode

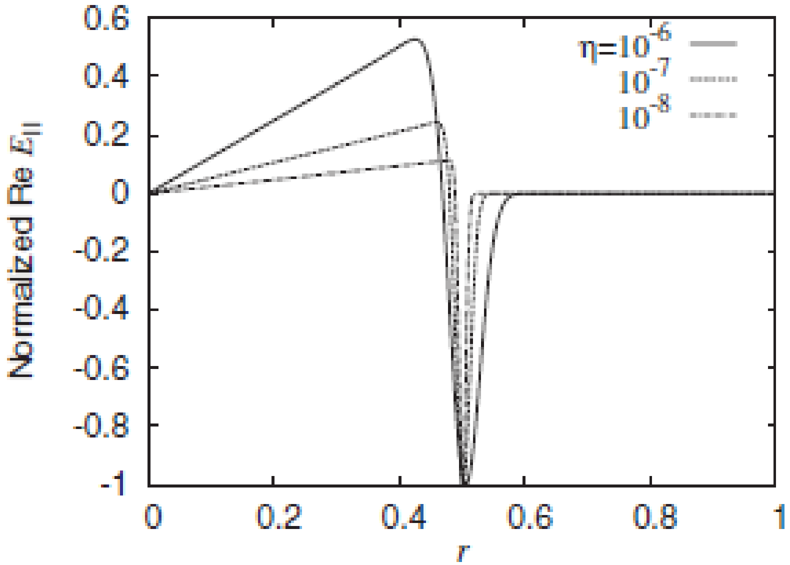
$\eta = 10^{-6}$



- Growth rate by the numerical matching technique agrees well with those obtained by global calculation (w/o matching)
- For zero beta, m/n=2/1 double tearing mode is unstable for $q_{\min} < 2$
- For finite beta, the m/n=2/1 mode becomes unstable even if it is non-resonant for $q_{\min} > 2$

Background: Correction of outer solution

- Considerably finite $E_{||}$ remains in outer region in m/n=1/1 internal kink mode
  - This situation simulates marginally stable state against ideal MHD, which includes high-beta toroidal plasmas



[M. Furukawa, S. Tokuda and L. –J. Zheng, Phys. Plasmas **17**, 052502 (2010).]

Non-zero $E_{||}$ in the $q < 1$ region indicates the relative importance of inertia and resistivity there

- **We have developed an ORDERING SCHEME in the outer region for high-beta RMHD, enabling us to obtain higher order corrections to the lowest-order outer solution**
  - note that the lowest-order outer solution satisfies $E_{||} = 0$

Significance of ordering scheme in outer region

- So far, traditional matched asymptotic expansion has used only the lowest-order outer solution, i.e. inertialess, ideal MHD

- As we will see, the order-by-order equations in the outer region form a hierarchy of generalized Newcomb equations, i.e.

$$\begin{cases} \mathcal{N}\psi_{(0)} = 0 \\ \mathcal{N}\psi_{(1)} = 0 \\ \mathcal{N}\psi_{(2)} = \underline{S(\psi_{(0)})} \\ \quad\vdots \end{cases}$$

$\mathcal{N}$ is the well-known Newcomb operator (second-order ordinary differential operator)

Inhomogeneous source term includes effects of inertia and resistivity

  - Applicable to plasmas marginally stable against ideal MHD
    - Inertia and resistivity play relatively important role even in outer region
  - We only need to solve second-order ODE for a scalar $\psi_{(j)}$
    - Very fast computation with less resource
  - Almost same module can be used in code development

- **Our new theory has therefore established a powerful solution method for boundary-layer problems**

[M. Furukawa and S. Tokuda, Phys. Plasmas **18**, 062502 (2011).]

Structure of solution method

- Divide the domain into OUTER and INNER regions

- Solve lowest-order Newcomb eq. in the outer region
- Solve RMHD in the inner region as a boundary-value problem with a guess eigenvalue

iteration

- Seeking a true eigenvalue by imposing continuity of lowest-order $\mathrm{d}\psi_{(0)}/\mathrm{d}r$ across the matching boundaries

Lowest-order solution

- First-order gives us a trivial solution, which can be included in the lowest-order solution

First-order solution

- Solve second-order Newcomb eq. in the outer region, which have inhomogeneous source terms
- Solve RMHD in the inner region as a boundary-value problem with a guess correction for eigenvalue

iteration

- Seeking a true correction for eigenvalue by imposing continuity of second-order $\mathrm{d}\psi_{(2)}/\mathrm{d}r$ across the matching boundaries

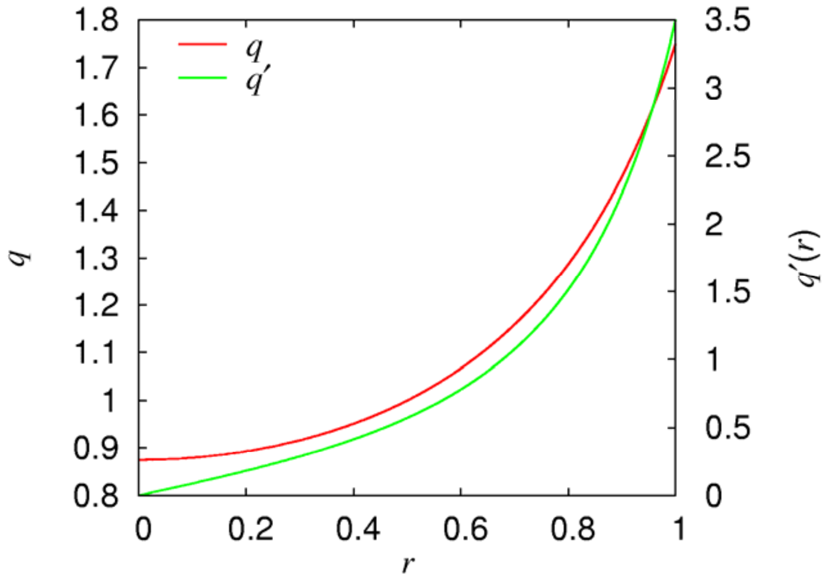
Second-order solution

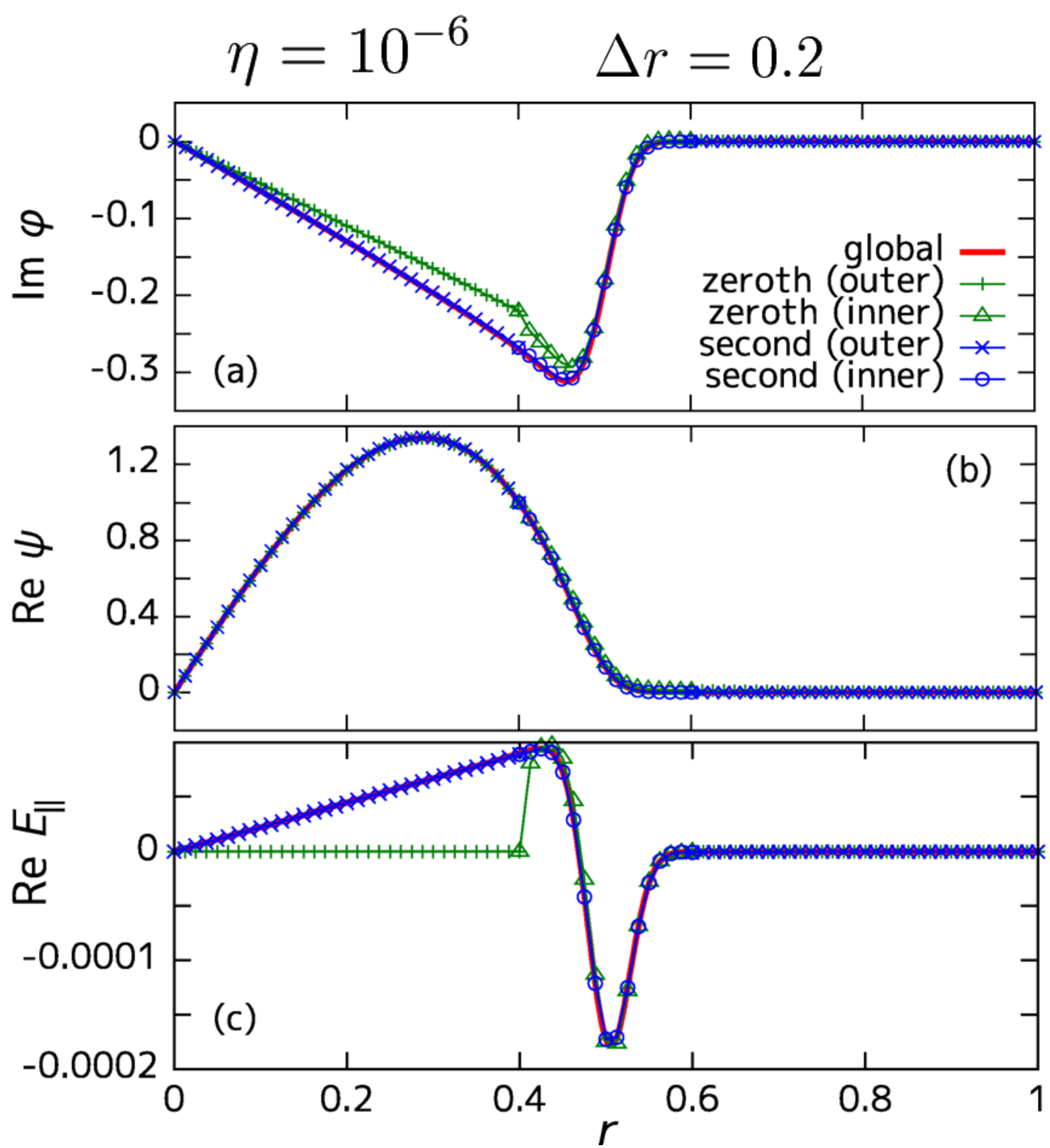
[M. Furukawa and S. Tokuda, Phys. Plasmas **18**, 062502 (2011).]

Example: Eigenfunction of m/n=1/1 internal kink mode

- Aspect ratio $A = 10$
- 800 grids in total
- $m/n = 1/1$ surface at $r = 0.5$
- Equilibrium current density:

$$j_\mathrm{t}(r) = j_{\mathrm{t}0}(1 - r^2)$$





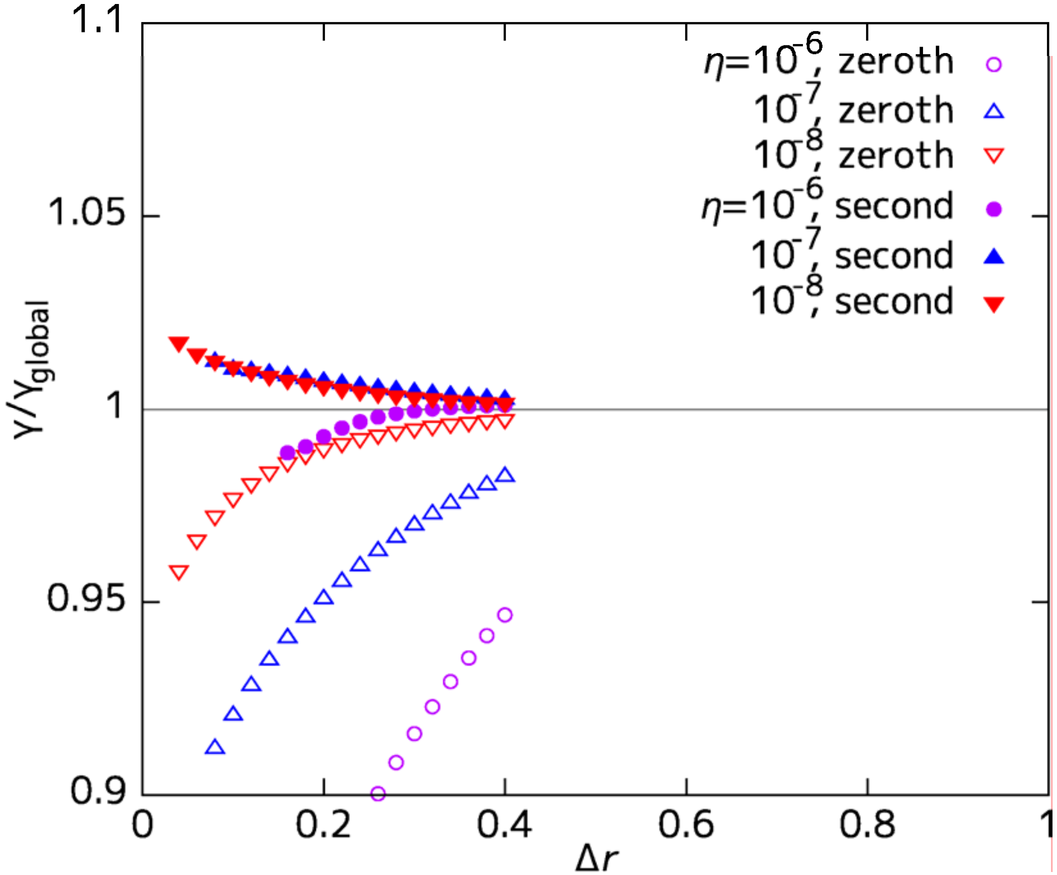
- **The finite $E_{||}$ in the outer region is successfully captured**

[M. Furukawa and S. Tokuda, Phys. Plasmas **18**, 062502 (2011).]

Example: Growth rate of m/n=1/1 internal kink mode

- Aspect ratio $A = 10$
- 800 grids in total
- $m/n = 1/1$ surface at $r = 0.5$
- Equilibrium current density:

$$j_{\mathrm{t}}(r) = j_{\mathrm{t0}}(1 - r^2)$$

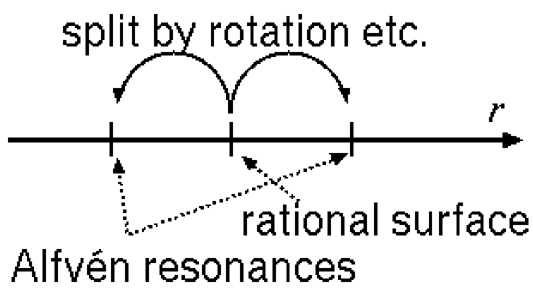


- **The accuracy of the growth rate improves by the correction of the outer solution**

[M. Furukawa and S. Tokuda, Phys. Plasmas **18**, 062502 (2011).]

Motivation: Extension to include FLR effects

- If we include FLR (Finite Larmor Radius) effects, or diamagnetic drift effects, mode can have finite real frequency, which makes
  - inertia important even in outer region
    - The higher-order correction to outer solution would be indispensable
  - Alfvén resonances split from rational surface
    - The finite-width inner region would be indispensable, because how far is the Alfven resonance from the rational surface would not be known prior to solving the problem; we do not know where we should put the inner layer in the conventional matched asymptotic expansion



- **In this presentation, I would like to focus on the development of the ordering scheme to solve the governing equations in the outer region order by order**

II. Ordering scheme in outer region including FLR effects

Review: High-beta reduced MHD

- Linearizing the high-beta reduced MHD equations, we obtain

[H. R. Strauss, Phys. Fluids **20**, 1354 (1977).]

$$\begin{cases} -\mathrm{i}\,\omega\nabla_\perp^2\varphi = -\mathrm{i}\,F\nabla_\perp^2\psi - \dfrac{\mathrm{i}\,mJ_0'}{r}\psi + \dfrac{\mathrm{i}\,m\kappa}{r}p \\ -\mathrm{i}\,\omega\psi = -\mathrm{i}\,F\varphi + \eta\nabla_\perp^2\psi \\ -\mathrm{i}\,\omega p = \dfrac{\mathrm{i}\,mp_0'}{r}\varphi \end{cases}$$

$$\left(\begin{aligned} &\boldsymbol{\kappa}\cdot\hat{\mathbf{r}} := \kappa = -B_{0\theta}^2/r \\ &F(r) := m\varepsilon\left(\frac{n}{m}+\frac{1}{q}\right) \\ &\mathbf{k} := m\nabla\theta - (n/R_0)\nabla z \end{aligned}\right)$$

where time dependence is assumed to be $\mathrm{e}^{-\mathrm{i}\,\omega t}$

- Introducing a small parameter $\epsilon \ll 1$, we found the following ordering scheme is appropriate:

(ϵ is not an inverse aspect ratio)

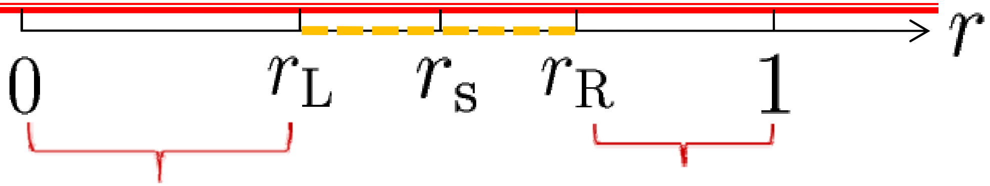
(Distinguish from equilibrium quantities with subscript $_0$)

$$\eta \sim \mathcal{O}(\epsilon^3)$$
$$\omega = \epsilon\omega_{(1)} + \epsilon^2\omega_{(2)} + \cdots$$
$$\varphi = \epsilon\varphi_{(1)} + \epsilon^2\varphi_{(2)} + \cdots$$
$$\psi = \psi_{(0)} + \epsilon\psi_{(1)} + \cdots$$
$$p = p_{(0)} + \epsilon p_{(1)} + \cdots$$
$$\boxed{\frac{\partial}{\partial r} \sim \mathcal{O}(1)}$$

These are similar to the ordering in the inner layer in the matched asymptotic expansion, except for this

[M. Furukawa and S. Tokuda, Phys. Plasmas **18**, 062502 (2011).]

Review: High-beta reduced MHD (cont'd)



- **Lowest order:**
  - $\mathcal{N}\psi_{(0)} = 0$

$$\mathcal{N} := \nabla_\perp^2 + \frac{m}{rF}\left(J_0' + \frac{m}{rF}\kappa p_0'\right)$$

$$\begin{cases} \varphi_{(1)} = \dfrac{\omega_{(1)}}{F}\psi_{(0)} \\ p_{(0)} = -\dfrac{m p_0'}{rF}\psi_{(0)} \end{cases}$$

- **First order:**
  - $\mathcal{N}\psi_{(1)} = 0$

$$\begin{cases} \varphi_{(2)} = \dfrac{1}{F}\left(\omega_{(1)}\psi_{(1)} + \omega_{(2)}\psi_{(0)}\right) \\ p_{(1)} = -\dfrac{m p_0'}{rF}\psi_{(1)} \end{cases}$$

- **Second order:**
  - $\mathcal{N}\psi_{(2)} = \underbrace{\frac{\omega_{(1)}}{F}\nabla_\perp^2\left(\frac{\omega_{(1)}}{F}\psi_{(0)}\right) + \frac{\mathrm{i}\,\eta}{\omega_{(1)}}\left(\frac{m}{rF}\right)^2\kappa p_0'\nabla_\perp^2\psi_{(0)}}$

the plasma inertia and resistivity come in this order as inhomogeneous terms

$$\begin{cases} \varphi_{(3)} = \dfrac{1}{F}\left(\omega_{(1)}\psi_{(2)} + \omega_{(2)}\psi_{(1)} + \omega_{(3)}\psi_{(0)} - \mathrm{i}\,\eta\nabla_\perp^2\psi_{(0)}\right) \\ p_{(2)} = -\dfrac{m p_0'}{rF}\left(\psi_{(2)} - \mathrm{i}\,\eta\nabla_\perp^2\psi_{(0)}\right) \end{cases}$$

[M. Furukawa and S. Tokuda, Phys. Plasmas **18**, 062502 (2011).]

Linearized Four-field model

[R. D. Hazeltine, M. Kotschenreuther, and P. J. Morrison, Phys. Fluids **28**, 2466 (1985).]

Vorticity equation

$$-\mathrm{i}\,\omega\nabla_\perp^2\varphi = -\frac{\mathrm{i}\,m}{r}\delta\beta_\mathrm{e}\frac{T_\mathrm{i}}{T_\mathrm{e}}\left[p_0'\nabla_\perp^2 + \left(p_0'' - \frac{p_0'}{r}\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\right]\varphi$$
$$-\mathrm{i}\,F\left(\nabla_\perp^2 + \frac{mJ_0'}{rF}\right)\psi + \frac{\mathrm{i}\,m}{r}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\kappa\beta_\mathrm{e}p$$

(β_e is explicitly written for clarity)

Parallel Ohm's law

$$-\mathrm{i}\,\omega\psi = -\mathrm{i}\,F\varphi + \left(\frac{\mathrm{i}\,m}{r}\delta\beta_\mathrm{e}p_0' + \eta\nabla_\perp^2\right)\psi + \delta\beta_\mathrm{e}\mathrm{i}\,Fp$$

Pressure equation (continuity equation)

$$-\mathrm{i}\,\omega\beta_\mathrm{e}p = \frac{\mathrm{i}\,m}{r}\left(\beta_\mathrm{e}p_0' - 2\beta\kappa\right)\varphi - 2\delta\beta\mathrm{i}\,F\left(\nabla_\perp^2 + \frac{mJ_0'}{rF}\right)\psi$$
$$+\left[\frac{\mathrm{i}\,m}{r}4\delta\kappa + \frac{1}{2}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\eta\nabla_\perp^2\right]\beta\beta_\mathrm{e}p - \beta\mathrm{i}\,Fv$$

Parallel equation of motion

$$-\mathrm{i}\,\omega v = -\frac{1}{2}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\frac{\mathrm{i}\,m}{r}\beta_\mathrm{e}p_0'\psi - \frac{1}{2}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\mathrm{i}\,F\beta_\mathrm{e}p$$
$$+\frac{\mathrm{i}\,m}{r}\delta\beta\frac{T_\mathrm{i}}{T_\mathrm{e}}\left[\frac{1}{2}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\beta_\mathrm{e}p_0' - 4\kappa\right]v$$

parallel wave number

$$F = k_\parallel := \varepsilon m\left(\frac{n}{m} + \frac{1}{q}\right)$$

parameter of FLR

$$\delta := \frac{1}{2\Omega\tau_\mathrm{A}}$$

$$\Omega := \frac{eB_0}{m_\mathrm{i}}$$

beta

$$\beta := \frac{\beta_\mathrm{e}}{1 + \frac{\beta_\mathrm{e}}{2}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)}$$

$$\beta_\mathrm{e} := \frac{2\mu_0 nT_\mathrm{e}}{B_0^2}$$

We would like to extend the ordering scheme suitable for this four-field model

Some considerations for natural extension of ordering (1)

[R. D. Hazeltine, M. Kotschenreuther, and P. J. Morrison, Phys. Fluids **28**, 2466 (1985).]

Vorticity equation

$$-\mathrm{i}\,\omega\nabla_\perp^2\varphi = -\frac{\mathrm{i}\,m}{r}\delta\beta_\mathrm{e}\frac{T_\mathrm{i}}{T_\mathrm{e}}\left[p_0'\nabla_\perp^2 + \left(p_0'' - \frac{p_0'}{r}\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\right]\varphi$$
$$-\mathrm{i}\,F\left(\nabla_\perp^2 + \frac{mJ_0'}{rF}\right)\psi + \frac{\mathrm{i}\,m}{r}\left(1+\frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\kappa\beta_\mathrm{e}p$$

These terms should be smaller than (at least) this term in order to obtain the well-known Newcomb equation in the lowest order

Parallel Ohm's law

$$-\mathrm{i}\,\omega\psi = -\mathrm{i}\,F\varphi + \left(\frac{\mathrm{i}\,m}{r}\delta\beta_\mathrm{e}p_0' + \eta\nabla_\perp^2\right)\psi + \delta\beta_\mathrm{e}\mathrm{i}\,Fp$$

Pressure equation (continuity equation)

$$-\mathrm{i}\,\omega\beta_\mathrm{e}p = \frac{\mathrm{i}\,m}{r}\left(\beta_\mathrm{e}p_0' - 2\beta\kappa\right)\varphi - 2\delta\beta\mathrm{i}\,F\left(\nabla_\perp^2 + \frac{mJ_0'}{rF}\right)\psi$$
$$+\left[\frac{\mathrm{i}\,m}{r}4\delta\kappa + \frac{1}{2}\left(1+\frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\eta\nabla_\perp^2\right]\beta\beta_\mathrm{e}p - \beta\mathrm{i}\,Fv$$

Parallel equation of motion

$$-\mathrm{i}\,\omega v = -\frac{1}{2}\left(1+\frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\frac{\mathrm{i}\,m}{r}\beta_\mathrm{e}p_0'\psi - \frac{1}{2}\left(1+\frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\mathrm{i}\,F\beta_\mathrm{e}p$$
$$+\frac{\mathrm{i}\,m}{r}\delta\beta\frac{T_\mathrm{i}}{T_\mathrm{e}}\left[\frac{1}{2}\left(1+\frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\beta_\mathrm{e}p_0' - 4\kappa\right]v$$

Some considerations for natural extension of ordering (2)

[R. D. Hazeltine, M. Kotschenreuther, and P. J. Morrison, Phys. Fluids **28**, 2466 (1985).]

Vorticity equation

$$-\mathrm{i}\,\omega\nabla_\perp^2\varphi = -\frac{\mathrm{i}\,m}{r}\delta\beta_\mathrm{e}\frac{T_\mathrm{i}}{T_\mathrm{e}}\left[p_0'\nabla_\perp^2 + \left(p_0'' - \frac{p_0'}{r}\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\right]\varphi$$
$$-\,\mathrm{i}\,F\left(\nabla_\perp^2 + \frac{mJ_0'}{rF}\right)\psi + \frac{\mathrm{i}\,m}{r}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\kappa\beta_\mathrm{e}p$$

Parallel Ohm's law

$$-\mathrm{i}\,\omega\psi = -\mathrm{i}\,F\varphi + \left(\frac{\mathrm{i}\,m}{r}\delta\beta_\mathrm{e}p_0' + \eta\nabla_\perp^2\right)\psi + \delta\beta_\mathrm{e}\mathrm{i}\,Fp$$

These terms should be smaller than these terms in order to obtain ideal Ohm's law in the lowest order

Pressure equation (continuity equation)

$$-\mathrm{i}\,\omega\beta_\mathrm{e}p = \frac{\mathrm{i}\,m}{r}\left(\beta_\mathrm{e}p_0' - 2\beta\kappa\right)\varphi - 2\delta\beta\mathrm{i}\,F\left(\nabla_\perp^2 + \frac{mJ_0'}{rF}\right)\psi$$
$$+\left[\frac{\mathrm{i}\,m}{r}4\delta\kappa + \frac{1}{2}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\eta\nabla_\perp^2\right]\beta\beta_\mathrm{e}p - \beta\mathrm{i}\,Fv$$

Parallel equation of motion

$$-\mathrm{i}\,\omega v = -\frac{1}{2}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\frac{\mathrm{i}\,m}{r}\beta_\mathrm{e}p_0'\psi - \frac{1}{2}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\mathrm{i}\,F\beta_\mathrm{e}p$$
$$+\frac{\mathrm{i}\,m}{r}\delta\beta\frac{T_\mathrm{i}}{T_\mathrm{e}}\left[\frac{1}{2}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\beta_\mathrm{e}p_0' - 4\kappa\right]v$$

Some considerations for natural extension of ordering (3)

[R. D. Hazeltine, M. Kotschenreuther, and P. J. Morrison, Phys. Fluids **28**, 2466 (1985).]

Vorticity equation

$$-\mathrm{i}\,\omega\nabla_\perp^2\varphi = -\frac{\mathrm{i}\,m}{r}\delta\beta_\mathrm{e}\frac{T_\mathrm{i}}{T_\mathrm{e}}\left[p_0'\nabla_\perp^2 + \left(p_0'' - \frac{p_0'}{r}\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\right]\varphi$$
$$- \mathrm{i}\,F\left(\nabla_\perp^2 + \frac{mJ_0'}{rF}\right)\psi + \frac{\mathrm{i}\,m}{r}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\kappa\beta_\mathrm{e}p$$

Parallel Ohm's law

$$-\mathrm{i}\,\omega\psi = -\mathrm{i}\,F\varphi + \left(\frac{\mathrm{i}\,m}{r}\delta\beta_\mathrm{e}p_0' + \eta\nabla_\perp^2\right)\psi + \delta\beta_\mathrm{e}\mathrm{i}\,Fp$$

Pressure equation (continuity equation)

$$-\mathrm{i}\,\omega\beta_\mathrm{e}p = \frac{\mathrm{i}\,m}{r}\left(\beta_\mathrm{e}p_0' - 2\beta\kappa\right)\varphi - 2\delta\beta\mathrm{i}\,F\left(\nabla_\perp^2 + \frac{mJ_0'}{rF}\right)\psi$$
$$+ \left[\frac{\mathrm{i}\,m}{r}4\delta\kappa + \frac{1}{2}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\eta\nabla_\perp^2\right]\beta\beta_\mathrm{e}p - \beta\mathrm{i}\,Fv$$

Parallel equation of motion

$$-\mathrm{i}\,\omega v = -\frac{1}{2}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\frac{\mathrm{i}\,m}{r}\beta_\mathrm{e}p_0'\psi - \frac{1}{2}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\mathrm{i}\,F\beta_\mathrm{e}p$$
$$+ \frac{\mathrm{i}\,m}{r}\delta\beta\frac{T_\mathrm{i}}{T_\mathrm{e}}\left[\frac{1}{2}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\beta_\mathrm{e}p_0' - 4\kappa\right]v$$

This term should be the same order as this term, which was assumed in deriving the four-field model

Some considerations for natural extension of ordering (4)

[R. D. Hazeltine, M. Kotschenreuther, and P. J. Morrison, Phys. Fluids **28**, 2466 (1985).]

Vorticity equation

$$-\mathrm{i}\,\omega\nabla_\perp^2\varphi = -\frac{\mathrm{i}\,m}{r}\delta\beta_\mathrm{e}\frac{T_\mathrm{i}}{T_\mathrm{e}}\left[p_0'\nabla_\perp^2 + \left(p_0'' - \frac{p_0'}{r}\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\right]\varphi$$
$$-\mathrm{i}\,F\left(\nabla_\perp^2 + \frac{mJ_0'}{rF}\right)\psi + \frac{\mathrm{i}\,m}{r}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\kappa\beta_\mathrm{e}p$$

Parallel Ohm's law

$$-\mathrm{i}\,\omega\psi = -\mathrm{i}\,F\varphi + \left(\frac{\mathrm{i}\,m}{r}\delta\beta_\mathrm{e}p_0' + \eta\nabla_\perp^2\right)\psi + \delta\beta_\mathrm{e}\mathrm{i}\,Fp$$

These terms represent diamagnetic drift frequency, which should be smaller than, or at most the same order as, the mode frequency

Pressure equation (continuity equation)

$$-\mathrm{i}\,\omega\beta_\mathrm{e}p = \frac{\mathrm{i}\,m}{r}\left(\beta_\mathrm{e}p_0' - 2\beta\kappa\right)\varphi - 2\delta\beta\mathrm{i}\,F\left(\nabla_\perp^2 + \frac{mJ_0'}{rF}\right)\psi$$
$$+\left[\frac{\mathrm{i}\,m}{r}4\delta\kappa + \frac{1}{2}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\eta\nabla_\perp^2\right]\beta\beta_\mathrm{e}p - \beta\mathrm{i}\,Fv$$

Parallel equation of motion

$$-\mathrm{i}\,\omega v = -\frac{1}{2}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\frac{\mathrm{i}\,m}{r}\beta_\mathrm{e}p_0'\psi - \frac{1}{2}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\mathrm{i}\,F\beta_\mathrm{e}p$$
$$+\frac{\mathrm{i}\,m}{r}\delta\beta\frac{T_\mathrm{i}}{T_\mathrm{e}}\left[\frac{1}{2}\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\beta_\mathrm{e}p_0' - 4\kappa\right]v$$

Natural extension of ordering in the outer region

- By using a small parameter $\epsilon \ll 1$, let us introduce the following ordering in the outer region:

(ϵ is not an inverse aspect ratio)

$$\eta \sim \mathcal{O}(\epsilon^3)$$
$$\omega = \epsilon\omega_{(1)} + \epsilon^2\omega_{(2)} + \cdots$$
$$\varphi = \epsilon\varphi_{(1)} + \epsilon^2\varphi_{(2)} + \cdots$$
$$\psi = \psi_{(0)} + \epsilon\psi_{(1)} + \cdots$$
$$p = p_{(0)} + \epsilon p_{(1)} + \cdots$$
$$\frac{\partial}{\partial r} \sim \mathcal{O}(1)$$

These have been adopted in Furukawa & Tokuda, PoP (2011)

and, for the new variable and the parameter,

$$v = \epsilon v_{(1)} + \epsilon^2 v_{(2)} + \cdots$$

This makes parallel and perpendicular inertia same order

$$\delta = \mathcal{O}(\epsilon)$$

By choosing β_{e} well smaller than unity, this makes the diamagnetic drift frequency smaller than the mode frequency

Problem stemming from sound wave

- The four-field model includes sound wave, which can be extracted by picking up corresponding terms as

$$\begin{cases} -\mathrm{i}\,\omega\beta_{\mathrm{e}}p = -\beta\mathrm{i}\,Fv \\ -\mathrm{i}\,\omega v = -\dfrac{1}{2}\left(1+\dfrac{T_{\mathrm{i}}}{T_{\mathrm{e}}}\right)\mathrm{i}\,F\beta_{\mathrm{e}}p \end{cases} \xrightarrow{\text{ordering}} \begin{cases} -\mathrm{i}\,\omega_{(1)}\beta_{\mathrm{e}}p_{(0)} = -\beta\mathrm{i}\,Fv_{(1)} \\ -\mathrm{i}\,\omega_{(1)}v_{(1)} = -\dfrac{1}{2}\left(1+\dfrac{T_{\mathrm{i}}}{T_{\mathrm{e}}}\right)\mathrm{i}\,F\beta_{\mathrm{e}}p_{(2)} \end{cases}$$

giving

$$\begin{aligned} p_{(2)} &= \frac{2\omega_{(1)}^2}{\beta\left(1+\frac{T_{\mathrm{i}}}{T_{\mathrm{e}}}\right)F^2}p_{(0)} \\ &= -\frac{2\omega_{(1)}^2 m p_0'}{\beta\left(1+\frac{T_{\mathrm{i}}}{T_{\mathrm{e}}}\right)rF^3}\psi_{(0)} \\ &\propto \frac{1}{\beta_{\mathrm{e}}}\psi_{(0)} \end{aligned}$$

I have used a relation similarly obtained by the ordering

c.f. vorticity eq.

$$-\mathrm{i}\,\omega\nabla_\perp^2\varphi = -\frac{\mathrm{i}\,m}{r}\delta\beta_{\mathrm{e}}\frac{T_{\mathrm{i}}}{T_{\mathrm{e}}}\left[p_0'\nabla_\perp^2 + \left(p_0'' - \frac{p_0'}{r}\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\right]\varphi - \mathrm{i}\,F\left(\nabla_\perp^2 + \frac{mJ_0'}{rF}\right)\psi + \frac{\mathrm{i}\,m}{r}\left(1+\frac{T_{\mathrm{i}}}{T_{\mathrm{e}}}\right)\kappa\beta_{\mathrm{e}}p$$

- The second-order vorticity equation, which leads to the Newcomb equation with inhomogeneous source terms, includes a term proportional to $\beta_{\mathrm{e}}p_{(2)}$
- This term should vanish when $\beta_{\mathrm{e}} \to 0$, however, it remains finite

How to resolve this problem?

Item 1: Retaining the sound wave somehow

By considering typical frequencies in the four-field model, we would develop an ordering scheme, including ordering for β_{e}

I would explain the idea below, however, this is still on going

Item 2: Decoupling the sound wave

The r.h.s. of the pressure equation in the four-field model originates from compressibility $-p\nabla \cdot \mathbf{v}$.
As described in Hazeltine PoF(85), the compressibility is actually a higher-order term in the inverse aspect ratio expansion, however, is retained by physical importance
Therefore, we would just neglect the compressibility in the pressure equation, according to the consistent ordering by the inverse aspect ratio

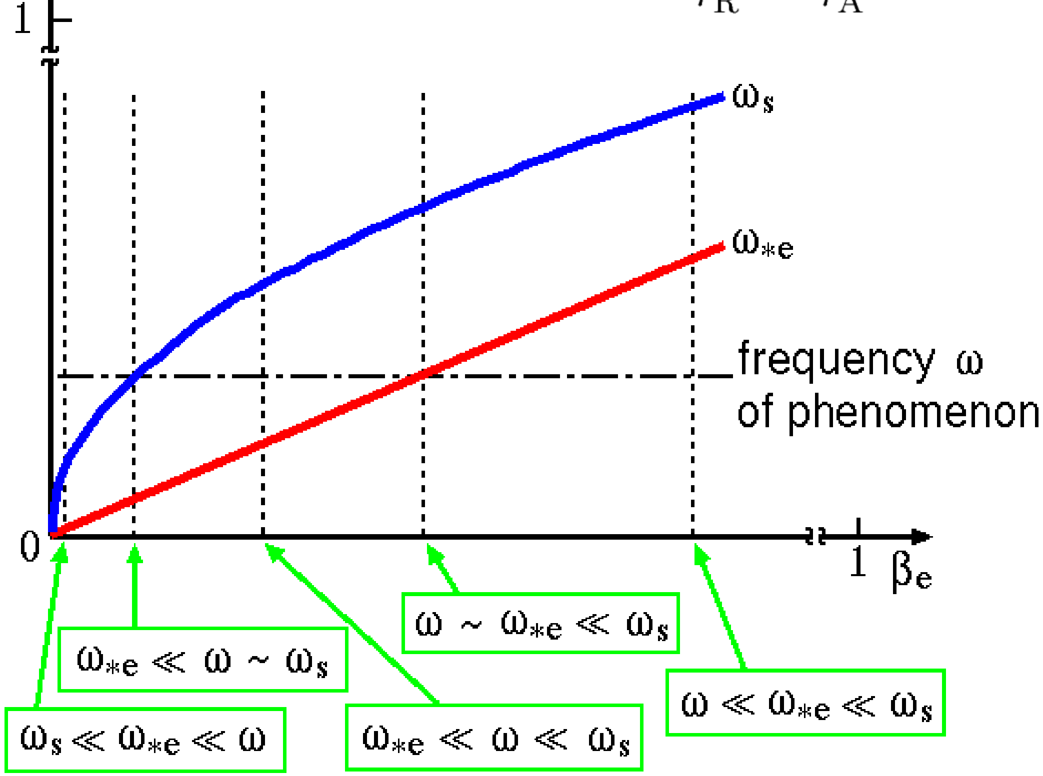
I have established an ordering scheme in the outer region including FLR effects

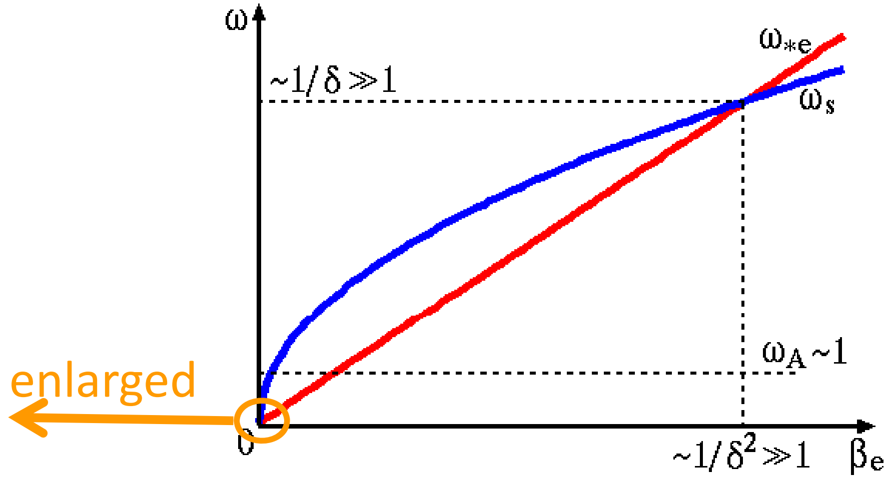
Item 1: Retaining sound wave

- Typical frequencies in the four-field model:
  - Alfvén frequency $\omega_{\mathrm{A}} = \pm F \sim 1$
  - diamagnetic drift frequency $\omega_{*\mathrm{e}} = \dfrac{m}{r}\delta\beta_{\mathrm{e}} p_0' \sim \delta\beta_{\mathrm{e}}$
  - sound wave frequency $\omega_{\mathrm{s}} = \pm F\sqrt{\dfrac{\beta}{2}\left(1+\dfrac{T_{\mathrm{i}}}{T_{\mathrm{e}}}\right)} \sim \sqrt{\beta_{\mathrm{e}}}$

  magnitude relation among these three would be important in the outer region

  - typical frequency in the inner layer $\dfrac{1}{T} \sim (F')^{2/3}\eta^{1/3}$
  - resistive diffusion $\dfrac{1}{\tau_{\mathrm{R}}} = \dfrac{\eta}{\tau_{\mathrm{A}}}$





Depending on the β_{e} value, the magnitude relation changes
Therefore we need an ordering also for β_{e}

<u>This is on going</u>

Item 2: Decoupling sound wave

- By dropping the $-p\nabla\cdot\mathbf{v}$ term, the linearized four-field model becomes

Vorticity equation

$$-\mathrm{i}\,\omega\nabla_\perp^2\varphi = -\frac{\mathrm{i}\,m}{r}\delta\beta_\mathrm{e}\frac{T_\mathrm{i}}{T_\mathrm{e}}\left[p_0'\nabla_\perp^2 + \left(p_0'' - \frac{p_0'}{r}\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\right]\varphi$$
$$-\,\mathrm{i}\,F\left(\nabla_\perp^2 + \frac{mJ_0'}{rF}\right)\psi + \frac{\mathrm{i}\,m}{r}\left(1+\frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\kappa\beta_\mathrm{e}p$$

Parallel Ohm's law

$$-\mathrm{i}\,\omega\psi = -\mathrm{i}\,F\varphi + \left(\frac{\mathrm{i}\,m}{r}\delta\beta_\mathrm{e}p_0' + \eta\nabla_\perp^2\right)\psi + \delta\beta_\mathrm{e}\mathrm{i}\,Fp$$

Pressure equation (continuity equation: just convection)

$$-\mathrm{i}\,\omega\beta_\mathrm{e}p = \frac{\mathrm{i}\,m}{r}\beta_\mathrm{e}p_0'\varphi$$

- In the following, the ordering scheme mentioned above is applied

Lowest order

- Writing the governing equations as

$$\begin{cases} -\mathrm{i}\,\omega\nabla_\perp^2\varphi = \epsilon C_{\varphi\varphi}(\varphi) + C_{\varphi\psi}(\psi) + C_{\varphi p}p \\ -\mathrm{i}\,\omega\psi = C_{\psi\varphi}\varphi + \left(\epsilon C_{\psi\psi}\psi + \epsilon^3 C_{\psi\eta}(\psi)\right) + \epsilon C_{\psi p}p \\ -\mathrm{i}\,\omega M_p p = C_{p\varphi}\varphi \end{cases}$$

the lowest-order equations are

$$\begin{cases} 0 = C_{\varphi\psi}(\psi_{(0)}) + C_{\varphi p}p_{(0)} \\ -\mathrm{i}\,\omega_{(1)}\psi_{(0)} = C_{\psi\varphi}\varphi_{(1)} + C_{\psi\psi}\psi_{(0)} + C_{\psi p}p_{(0)} \\ -\mathrm{i}\,\omega_{(1)}M_p p_{(0)} = C_{p\varphi}\varphi_{(1)} \end{cases}$$

These give us

$$\varphi_{(1)} = \frac{\omega_{(1)}}{F}\psi_{(0)}$$

$$p_{(0)} = -\frac{mp_0'}{rF}\psi_{(0)}$$

- The vorticity equation leads to the lowest-order Newcomb equation as

$$\mathcal{N}\psi_{(0)} = 0 \qquad \mathcal{N} := \nabla_\perp^2 + \frac{m}{rF}\left[J_0' + \frac{m}{rF}\kappa\beta_\mathrm{e}p_0'\left(1 + \frac{T_\mathrm{i}}{T_\mathrm{e}}\right)\right]$$

First order

$$\begin{cases} 0 = C_{\varphi\psi}(\psi_{(1)}) + C_{\varphi p} p_{(1)} \\ -\mathrm{i}\,\omega_{(2)}\psi_{(0)} - \mathrm{i}\,\omega_{(1)}\psi_{(1)} = C_{\psi\varphi}\varphi_{(2)} + C_{\psi\psi}\psi_{(1)} + C_{\psi p} p_{(1)} \\ -\mathrm{i}\,\omega_{(2)} M_p p_{(0)} - \mathrm{i}\,\omega_{(1)} M_p p_{(1)} = C_{p\varphi}\varphi_{(2)} \end{cases}$$

gives us

$$\varphi_{(2)} = \frac{1}{F}\left(\omega_{(1)}\psi_{(1)} + \omega_{(2)}\psi_{(0)}\right)$$

$$p_{(1)} = -\frac{m p_0'}{rF}\psi_{(1)}$$

and the first-order Newcomb equation as

$$\mathcal{N}\psi_{(1)} = 0$$

(with the same operator $\mathcal{N}$ as the lowest order)

Second order

- Similarly, we obtain

$$\varphi_{(3)} = \frac{1}{F}\left(\omega_{(1)}\psi_{(2)} + \omega_{(2)}\psi_{(1)} + \omega_{(3)}\psi_{(0)}\right) + \frac{1}{\mathrm{i}\,F}\frac{\omega_{(1)}}{\omega_{(1)} + \frac{m}{r}\delta\beta_{\mathrm{e}}p_0'}\eta\nabla_\perp^2\psi_{(0)}$$

$$p_{(2)} = -\frac{mp_0'}{rF}\left(\psi_{(2)} - \frac{\mathrm{i}}{\omega_{(1)} + \frac{m}{r}\delta\beta_{\mathrm{e}}p_0'}\eta\nabla_\perp^2\psi_{(0)}\right)$$

and the second-order Newcomb equation as

$$\mathcal{N}\psi_{(2)} = \boxed{\frac{1}{F}\left(\omega_{(1)} - \frac{m}{r}\delta\beta_{\mathrm{e}}p_0'\frac{T_{\mathrm{i}}}{T_{\mathrm{e}}}\right)\nabla_\perp^2\left(\frac{\omega_{(1)}}{F}\psi_{(0)}\right) - \frac{m}{rF}\delta\beta_{\mathrm{e}}\frac{T_{\mathrm{i}}}{T_{\mathrm{e}}}\left(p_0'' - \frac{p_0'}{r}\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\left(\frac{\omega_{(1)}}{F}\psi_{(0)}\right) + \frac{\mathrm{i}}{\omega_{(1)} + \frac{m}{r}\delta\beta_{\mathrm{e}}p_0'}\left(\frac{m}{rF}\right)^2\kappa\beta_{\mathrm{e}}p_0'\left(1 + \frac{T_{\mathrm{i}}}{T_{\mathrm{e}}}\right)\eta\nabla_\perp^2\psi_{(0)}}$$

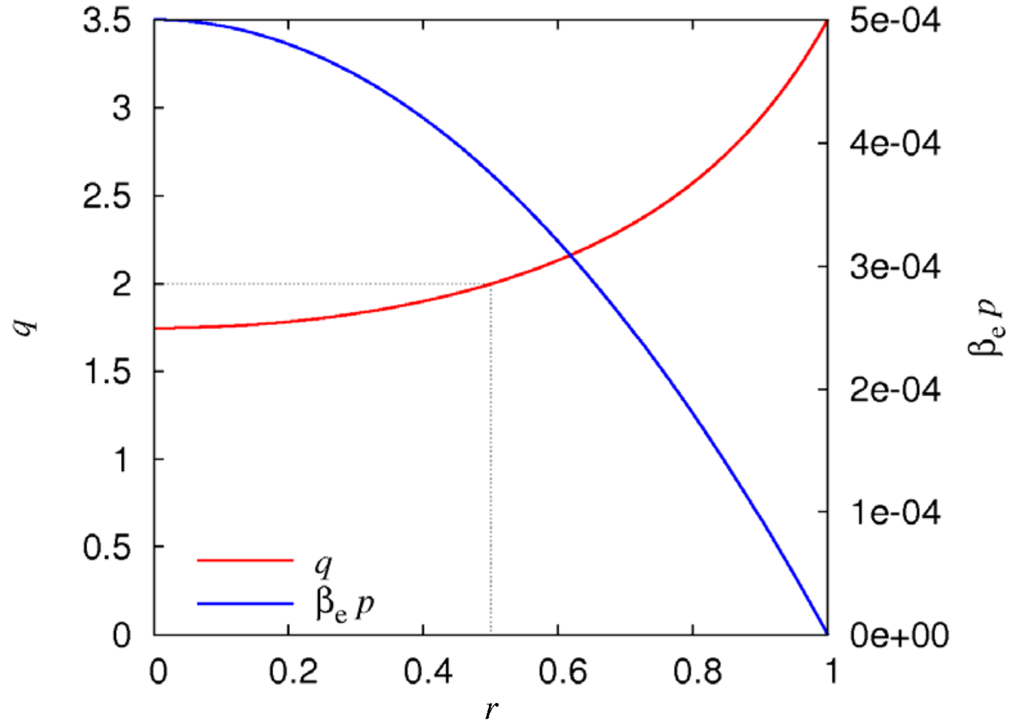
inhomogeneous source terms, which make the second-order correction to the outer solution

(with the same operator $\mathcal{N}$ as the lowest order)

III. Numerical results

m/n=2/1 tearing mode with FLR effects

- Aspect ratio $A = 10$
- 200 grids are used in the whole domain

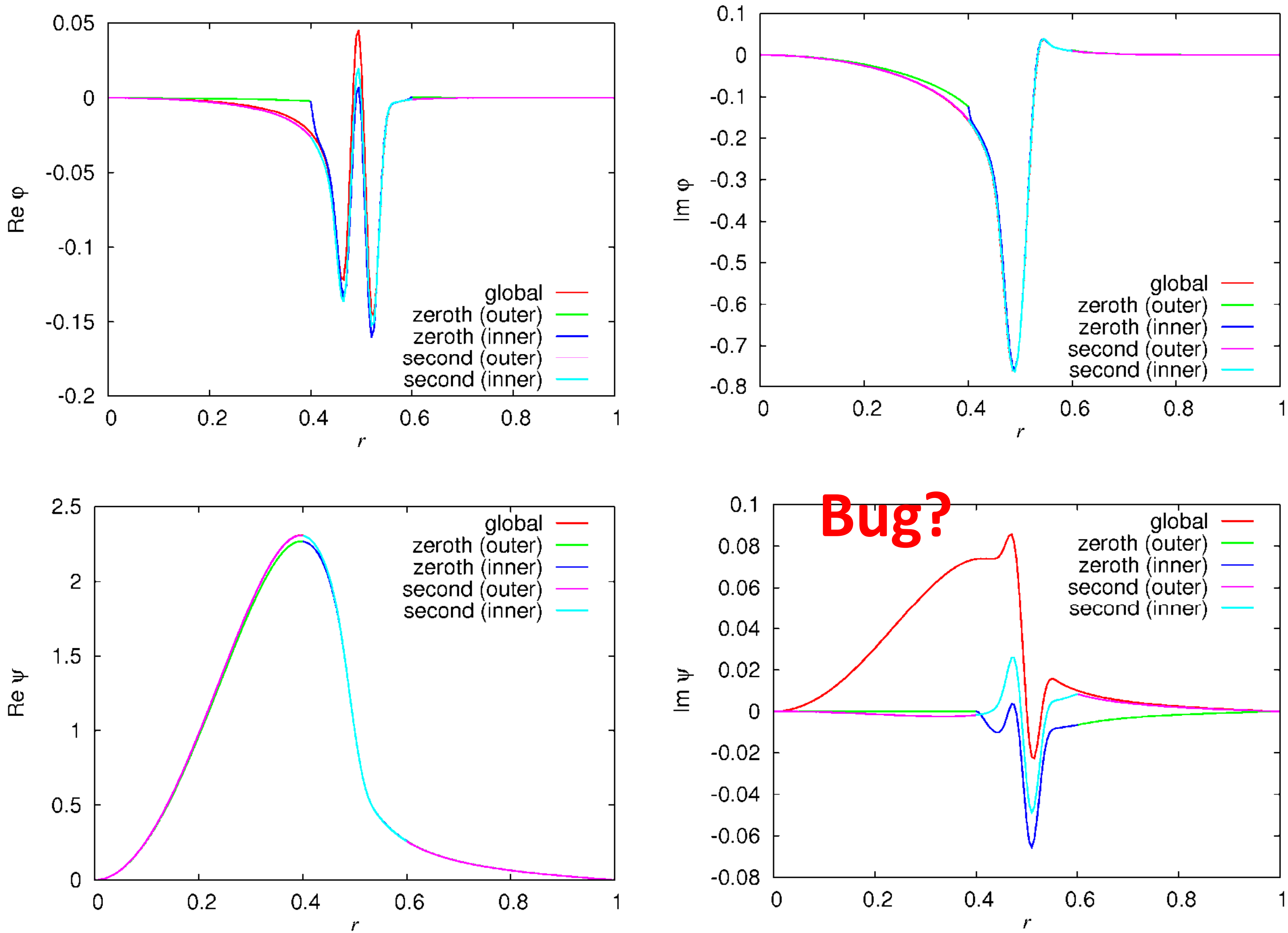


$$\beta_\mathrm{e} = 5 \times 10^{-4}$$

$$\frac{T_\mathrm{i}}{T_\mathrm{e}} = 1$$

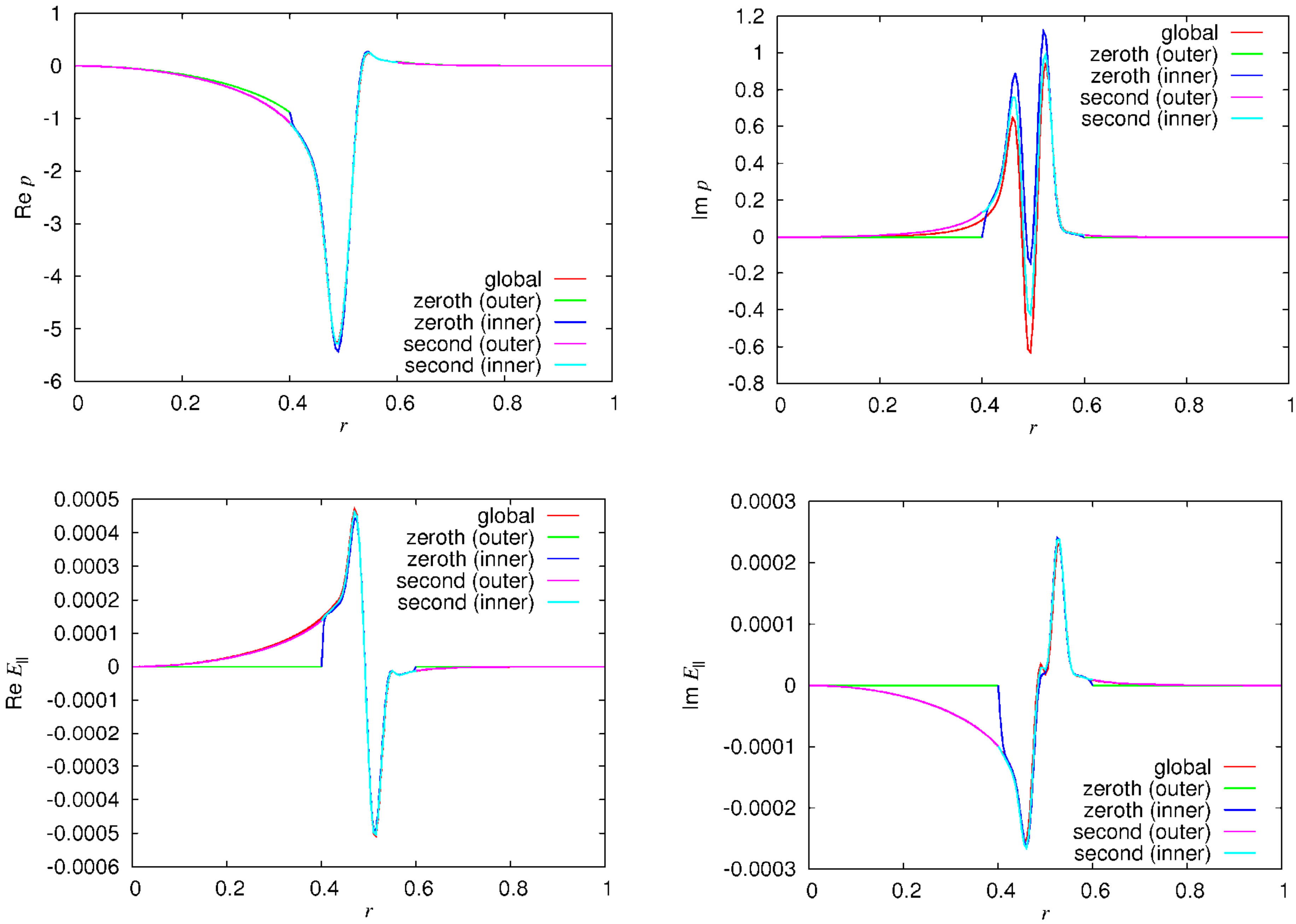
- Eigenvalue for $\delta = 0.1$, $\eta = 10^{-6}$
  - w/o matching (global) $\omega = 1.8272 \times 10^{-5} + 2.8913 \times 10^{-4}\mathrm{i}$
  - w/ matching ($\Delta r = 0.2$) $\omega = 1.5483 \times 10^{-5} + 2.8862 \times 10^{-4}\mathrm{i}$

m/n=2/1 tearing mode with FLR effects (eigenfunction 1)



Bug?

m/n=2/1 tearing mode with FLR effects (eigenfunction 2)



Conclusions

- We have extended an ordering scheme in outer region to include FLR effects
  - In this case the compressibility was neglected according to the consistent inverse aspect ratio ordering

- We have obtained some numerical results, which seem to show success of the extended ordering scheme

- We have also found a direction how to include compressibility in the ordering scheme
  - We need to specify an ordering for β_{e} , since the magnitude relation among typical frequencies in the four-field model changes depending on β_{e}