Analytical and Numerical Studies on Acceleration Phase of Collisionless Magnetic Reconnection

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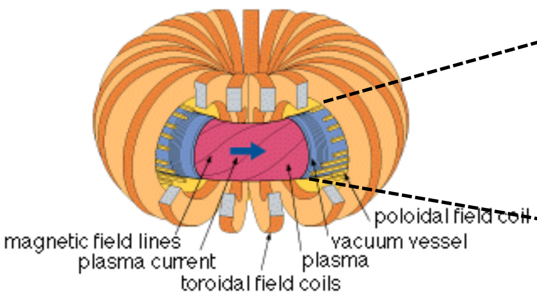
Introduction

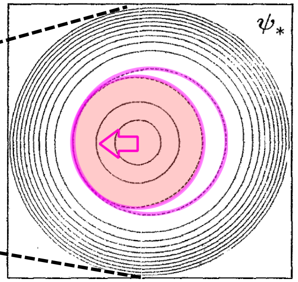
- Magnetic reconnection is triggered by dissipation/microscopic effects.

  (singular perturbations of ideal MHD)

- If plasma is either collisionless ($R_m \sim 10^{9-13}$) or close to the ideal MHD stability limit ($\Delta' \sim \infty$), the resistive MHD theory cannot explain the observed reconnection speeds. $\Rightarrow$ collisionless magnetic reconnection

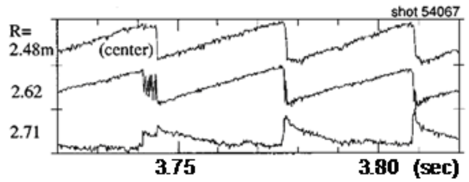
Tokamak





Internal kink-tearing mode
(from Biskamp 1991)

Nonlinear evolution



Sawtooth oscillations
(collapse faster than predicted by Resistive MHD model)

· Numerical simulations show acceleration of collisionless reconnection in nonliear phase

[Ottaviani and Porceli, PRL (1993), Matsumoto *et al.*, PoP (2005).]

· However, conventional methods (such as asymptotic matching and perturbation expansion) have difficulty in analysing the nonlinear evolution.

- We take a new theoretical approach based on variational principle in order to clarify the acceleration mechanism. Our analytical prediction is also verified by using a direct numerical simulation.

Triggers of reconnection in two-fluid model

Faraday's law		Generalized Ohm's law
$\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}$	$\Leftarrow$	$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} + \underbrace{\frac{d_i}{n}(\boldsymbol{j} \times \boldsymbol{B} - \nabla p_e)}_{(1)} + \underbrace{\frac{d_e^2}{n}\frac{d\boldsymbol{j}}{dt}}_{(2)} + \underbrace{\eta \boldsymbol{j}}_{(3)} - \underbrace{\eta_2 \nabla^2 \boldsymbol{j}}_{(4)}$

$\boldsymbol{v}$: ion velocity, n: number density, $\boldsymbol{j}$: current, p_e: electron pressure

(1). Hall effect: $d_i = (\text{ion skin depth})/L$

$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{v}_e \times \boldsymbol{B})$ where $\boldsymbol{v}_e = \boldsymbol{v} - d_i \boldsymbol{j}/n$ $\cdots$ no reconnection, by itself

(2). Electron inertia: $d_e = (\text{electron skin depth})/L$ $\cdots$ collisionless reconnection

(3). Resistivity: η $\cdots$ collisional reconnection

Ref. Rutherford theory (linear phase $\propto e^{\gamma t} \Rightarrow$ nonlinear phase $\propto t$)

(4). Electron viscosity: η_2 $\cdots$ collisional reconnection

In large tokamaks, $\boxed{(1) \gg (2) \gtrsim (3) \gg (4)}$

☞ We will focus on electron inertia (2) and study nonlinear acceleration mechanism of collisionless reconnection.

Analytical model of this work

2D MHD model with electron inertia

For $\boldsymbol{v} = \nabla\phi(x,y,t) \times \boldsymbol{e}_z$ and $\boldsymbol{B} = \nabla\psi(x,y,t) \times \boldsymbol{e}_z$,

$$\text{Vorticity equation:} \quad \frac{\partial \nabla^2\phi}{\partial t} - [\phi, \nabla^2\phi] - [\nabla^2\psi, \psi] = 0, \tag{1}$$

$$\text{(Collisionless) Ohm's law:} \quad \frac{\partial(\psi - d_e^2\nabla^2\psi)}{\partial t} - [\phi, \psi - d_e^2\nabla^2\psi] = 0, \tag{2}$$

where $d_e(\ll L)$: electron skin depth, and $[f,g] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}$.

This is known as a Hamiltonian system. (no dissipation)

- Hamiltonian: $H = \dfrac{1}{2}\displaystyle\int d^2x \left[|\nabla\phi|^2 + |\nabla\psi|^2 + d_e^2(\nabla^2\psi)^2\right]$
- Ohm's law $(2) \Leftrightarrow \partial_t\psi_e + \boldsymbol{v}\cdot\nabla\psi_e = 0$

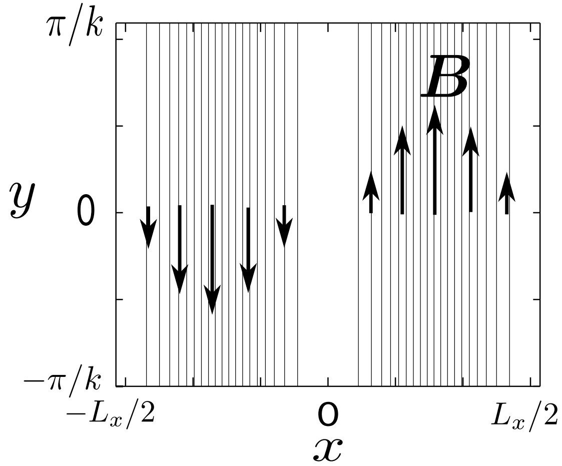
  Instead of magnetic flux ψ, electron's canonical momentum $\psi_e = \psi - d_e^2\nabla^2\psi$ is the frozen-in flux.

  $\Rightarrow$ Reconnection is possible without any dissipation mechanism.

We consider

1D equilibrium (periodic in both x and y directions)

$$\phi \equiv 0 \text{ (no flow)}, \quad \psi(x) = \cos\frac{2\pi x}{L_x} \quad \text{on } \left[-\frac{L_x}{2}, \frac{L_x}{2}\right]$$



- Collisionless magnetic reconnection spontaneously occurs at resonant surfaces $x = 0, \pm L_x/2$.
- For sufficiently small wavenumber k in the y direction, this instability ($\Delta' \sim \infty$) is similar to the $m = 1$ kink-tearing mode in tokamaks.

▶ The reconnection process mainly leads to the following energy conversion;

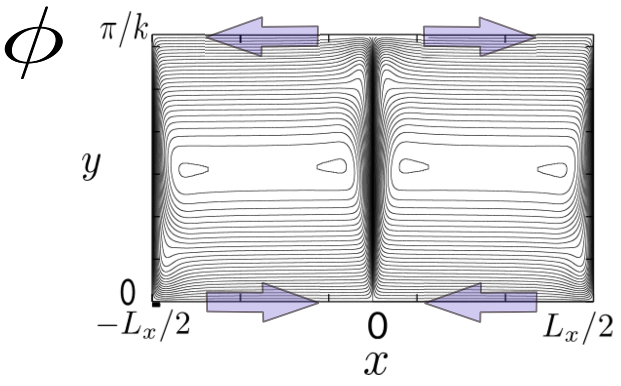
$$\boxed{\frac{1}{2}\int |\nabla\psi|^2 d^2x} \xrightarrow{\text{Relaxation}} \boxed{\frac{1}{2}\int |\nabla\phi|^2 d^2x \text{ and } \frac{1}{2}\int d_e^2 J^2 d^2x} \quad (J = -\nabla^2\psi)$$

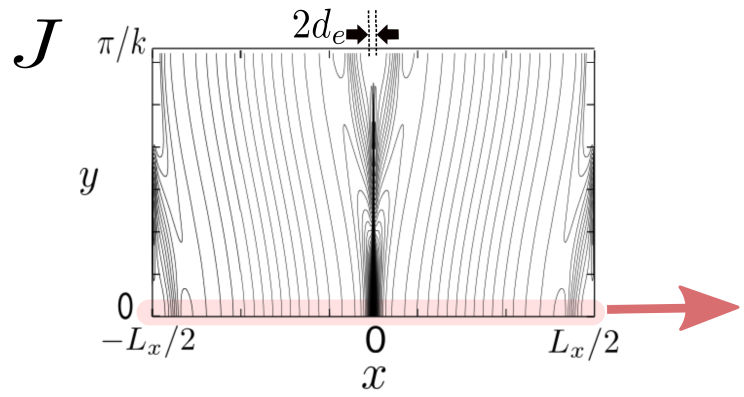
Direct numerical simulation

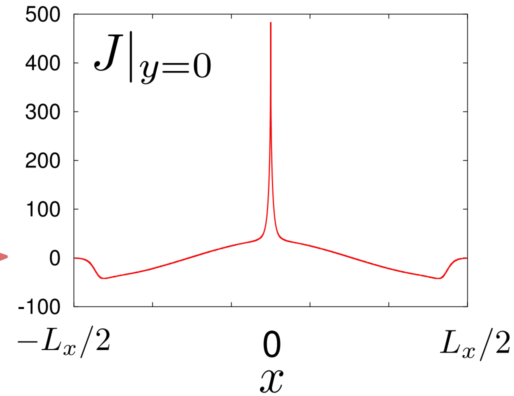
[Finite difference method in x direction ($\sim$10,000 grids), Spectral method in y direction ($\sim$100 modes)]

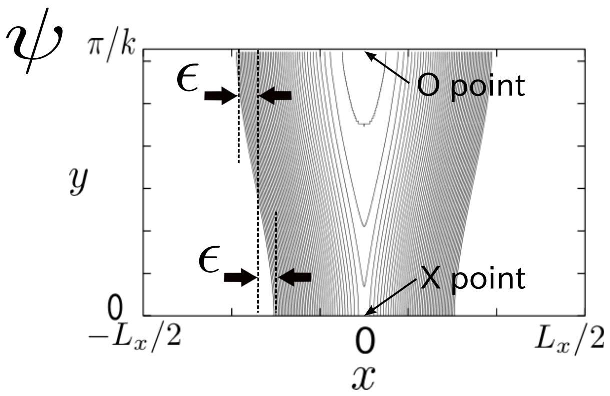
Define ϵ as maximum displacement in x direction ($\approx$ half widht of magnetic island).

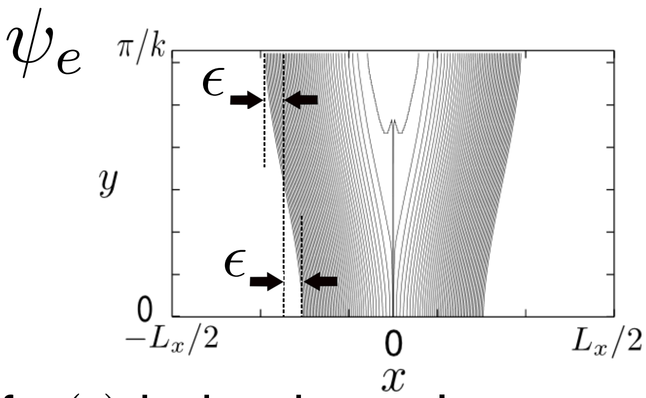
$\boxed{\text{Snapshots of contours when } \epsilon = 4.2 d_e}$ $(d_e/L_x = 0.01,\ k = 0.5/L_x)$







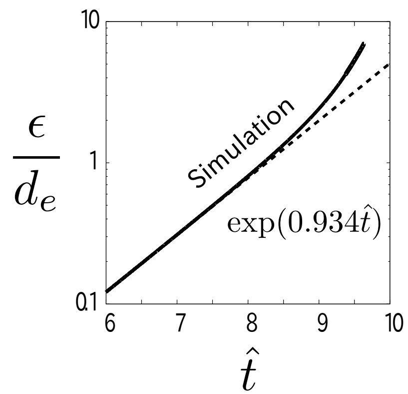




Strong currect spike develops
($\Rightarrow$ Simulation terminates due
to the limit of resolution.)

➡ Growth of $\epsilon(t)$ indeed accelerates when it exceeds the electron skin depth d_e.

[Ottaviani and Porceli, PRL (1993)]



Construction of variational principle

- Perturbations, $(0, \psi_e) \to (\tilde{\phi}, \tilde{\psi}_e)$, that preserve the flux ψ_e can be generated by a function $G(x, y, t)$ such that

$$\boxed{\begin{aligned}&\tilde{\phi}(x + \partial_y G(x, y, t), y, t) = \partial_t G(x, y, t),\\ &\tilde{\psi}_e(x + \partial_y G(x, y, t), y, t) = \psi_e(x)\end{aligned}}$$

$\Rightarrow$ Ohm's law (2) is solved! (which is built-in as a constraint on $\tilde{\phi}$ and $\tilde{\psi}_e$)

- Lagrangian: $L[G] = \frac{1}{2} \int \left(|\nabla \tilde{\phi}|^2 \underbrace{- |\nabla \tilde{\psi}|^2 - d_e^2 |\nabla^2 \tilde{\psi}|^2}_{\text{This play a role of potential energy}} \right) d^2x = K - W$

Variational principle: $\delta \int L[G] dt = 0$ w.r.t. $\forall \delta G$ $\Rightarrow$ Vorticity eq. (1)

> If the potential energy decreases ($\delta W < 0$) for some function G, then such a perturbation will grow with the release of free energy.

(The MHD energy principle is extended to two-fluid model.)

Linear stability analysis ($\epsilon \ll d_e$)

Small-amplitude expansion ($|G| \sim \epsilon \ll d_e$) around equilibrium state

$$L(\tilde{\phi}, \tilde{\psi}_e) = L(\psi_e) + \underbrace{L^{(1)}(\psi_e; G)}_{0 \text{ at equilibrium}} + \frac{1}{2} L^{(2)}(\psi_e; G, G) + \frac{1}{6} L^{(3)}(\psi_e; G, G, G) + \dots$$

- The 2nd-order Lagrangian $L^{(2)}$ governs the linearized dynamics.

  $\Rightarrow$ By putting $G(x, y, t) = \epsilon(t)\hat{\xi}(x)\dfrac{\sin ky}{k}$ with $\epsilon(t) \propto e^{\gamma t}$, we obtain

  Eigenvalue problem (4th order ODE)

  $$-\left\{\left[(\gamma/k)^2 + (\psi_e')^2\right]\hat{\xi}'\right\}' + k^2\left[(\gamma/k)^2 + (\psi_e')^2\right]\hat{\xi} = d_e^2 \psi_e' J''' \hat{\xi} + \psi_e' d_e^2 \nabla^2 (1 - d_e^2 \nabla^2)^{-1} \nabla^2 (\psi_e' \hat{\xi})$$

  (the prime $'$ denotes x derivative.)

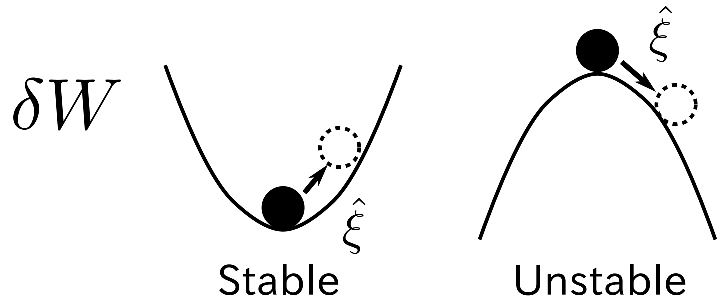
- Around marginal stability $\gamma \sim 0$, the boundary layers exist at positions where $\psi_e' = 0$.

  For $\gamma \neq 0$ and $d_e \neq 0$, the eigenfunctions $\hat{\xi}$ must be regular.

  (The MHD singularity is removed by the electron inertia.)

Energy principle for linear stability

$$\boxed{-\gamma^2\delta I = \delta W}$$ (⇐ Eigenvalue problem)



$$\delta I = \int dx \frac{1}{k^2}\left(|\hat{\xi}'|^2 + k^2|\hat{\xi}|^2\right) > 0$$

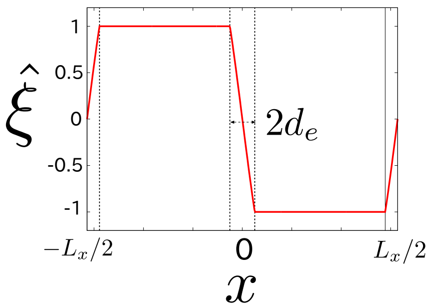
$$\delta W = \int dx \Big[\underbrace{|\nabla(\psi_e'\hat{\xi})|^2}_{>0} + \underbrace{\psi_e'\psi'''|\hat{\xi}|^2}_{<0} - \underbrace{\nabla^2(\psi_e'\hat{\xi}^*)d_e^2(1-d_e^2\nabla^2)^{-1}\nabla^2(\psi_e'\hat{\xi})}_{<0}\Big]$$

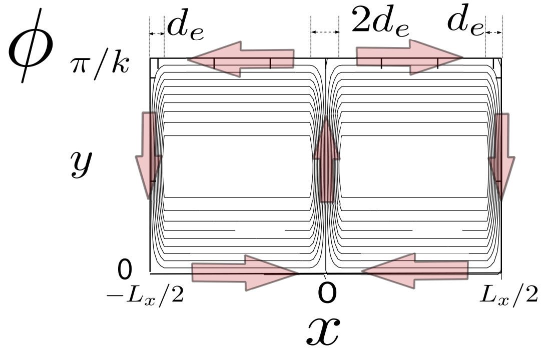
(i) magnetic field tension (ii) magnetic shear (iii) electron inertia

- (i)+(ii) > 0 $\Rightarrow$ Stable $\delta W > 0$ in the MHD limit $d_e = 0$
- (i)+(iii) > 0 $\Rightarrow$ Stable $\delta W > 0$ without the magnetic shear (or current)
  (The effect of electron inertia weakens the magnetic field tension only in the small scale $\sim d_e$)

Test function that makes δW negative

- Let us choose the following piecewise-linear function.





$$\left(\begin{array}{l} G(x,y,t) = \epsilon(t)\hat{\xi}(x)\dfrac{\sin ky}{k} \\ \phi(x,y,t) = \partial_t G \end{array} \right)$$

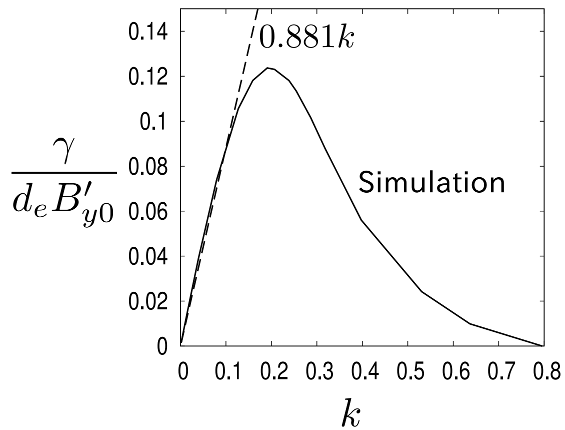
- Then, the 2nd-order Lagrangian is reduced to

$$L^{(2)}(\hat{\epsilon}) \simeq \frac{2\pi}{k} B_{y0}'^2 d_e^3 \left[\left(\frac{d\hat{\epsilon}}{d\hat{t}} \right)^2 - U(\hat{\epsilon}) \right] \quad \text{where} \quad \begin{cases} \hat{\epsilon} = \epsilon/d_e, \quad \hat{t} = t/\tau_e, \\ \tau_e^{-1} = d_e k B_{y0}' \end{cases}$$

Potential energy: $U(\hat{\epsilon}) = -\frac{1+27e^{-2}}{6}\hat{\epsilon}^2 = -0.776\hat{\epsilon}^2$

$\Rightarrow$ Linear growth rate: $\gamma = \sqrt{0.776}/\tau_e = 0.881/\tau_e$

This agrees with the results of conventinal asymptotic matching method as well as our numerical simulation.



Nonlinear stability analysis ($\epsilon > d_e$)

Remark: Failure of perturbation analysis

Let us try to continue the perturbation expansion of Lagrangian.

Nonlinear perturbations
$$\boxed{\begin{aligned}&\tilde{\phi}(x+\partial_y G(x,y,t),y,t) = \partial_t G(x,y,t),\\ &\tilde{\psi}_e(x+\partial_y G(x,y,t),y,t) = \psi_e(x)\end{aligned}}$$

$$\Rightarrow \quad \boxed{\begin{aligned}\tilde{\phi} &= G_t - G_y G_t' + \frac{1}{2}(G_y^2 G_t')' - \frac{1}{6}(G_y^3 G_t')'' + \frac{1}{24}(G_y^4 G_t')''' + O(\epsilon^6),\\ \tilde{\psi}_e &= \psi_e - G_y\psi_e' + \frac{1}{2}(G_y^2\psi_e')' - \frac{1}{6}(G_y^3\psi_e')'' + \frac{1}{24}(G_y^4\psi_e')''' + O(\epsilon^5),\end{aligned}}$$

where $G_t = \partial_t G$, $G_y = \partial_y G$.

However, the linearly unstable mode has a steep gradient, $G' \sim G/d_e$.

$\Rightarrow$ The above expansion fails to converge when $\epsilon = \max|G_y| \to d_e$.
(In fact, we will find that ϵ easily exceeds d_e.)

For $\epsilon > d_e$, full-nonlinear analysis is required around the inner layers.

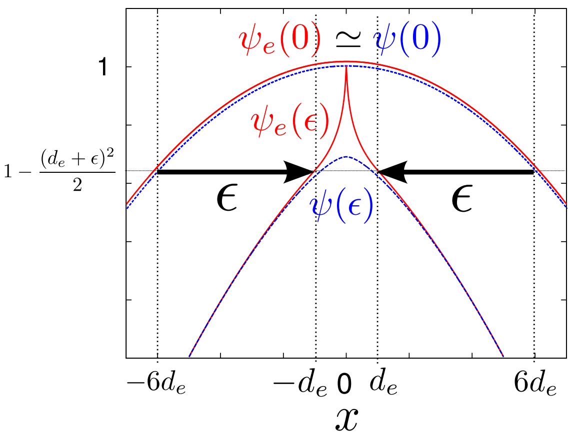
Potential energy change "around the X-point"

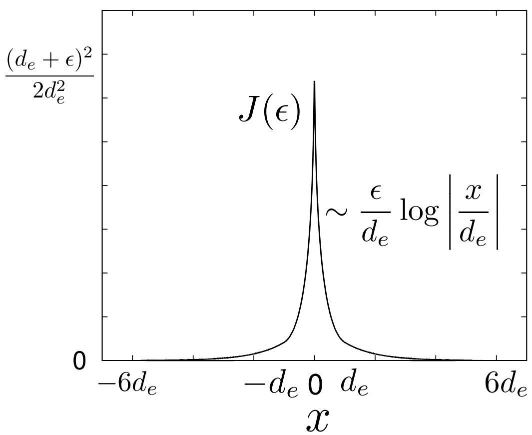
We have directly imposed a nonlinear displacement $\epsilon > d_e$ and investigated subsequent potential energy change.

> Around the X point, decrease of potential energy is found to be steeper than that in the linear regime

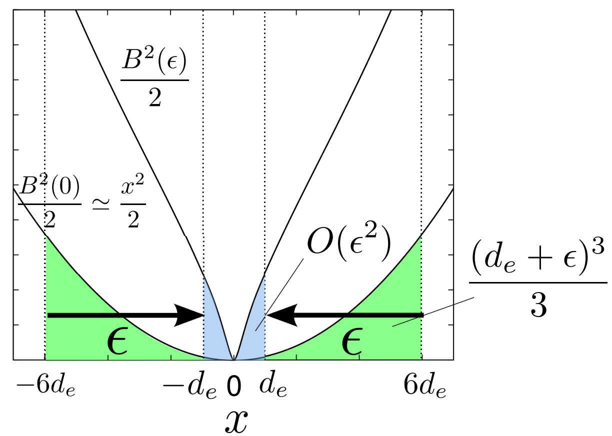
- Around the X point, ψ_e is compressed by the inflow.
- By this convection, outer region loses magnetic energy of $O(\epsilon^3)$, but inner layer gains magnetic and current energy, at most, of $O(\epsilon^2)$. $\Rightarrow$ Potential decreases in ϵ^3

(When $\epsilon = 5d_e$)





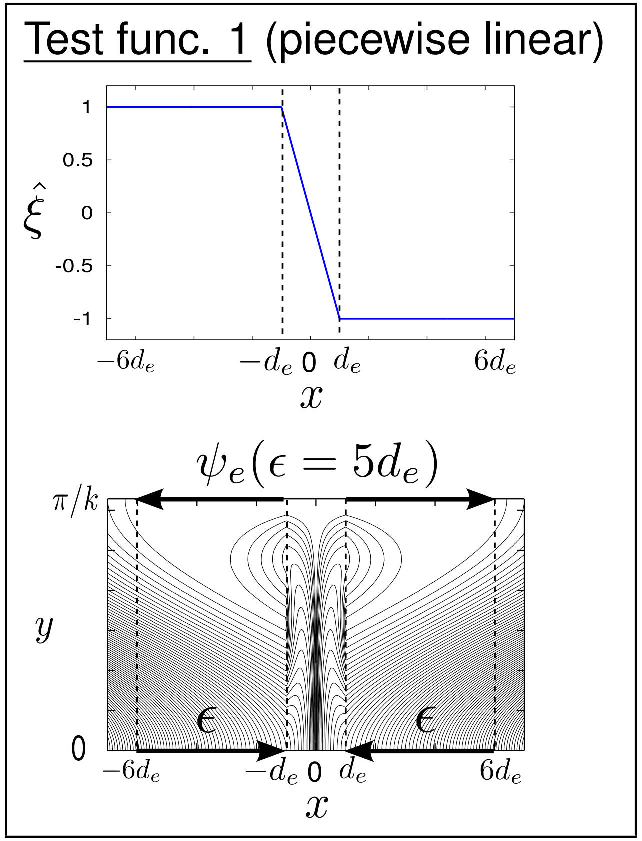
$$\Rightarrow \boxed{\delta \int d_e^2 \frac{J^2}{2} dx = O(\epsilon^2)}$$

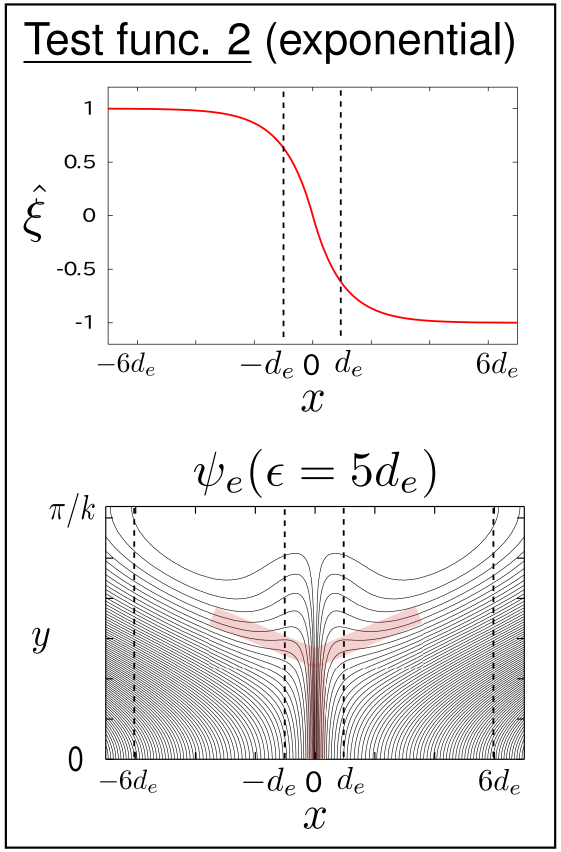


$$\Rightarrow \boxed{\delta \int \frac{B^2}{2} dx = -\frac{\epsilon^3}{3} + O(\epsilon^2)}$$

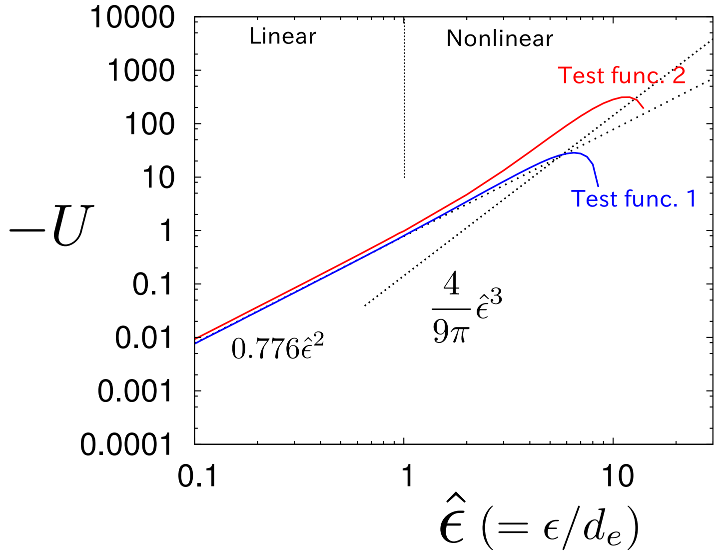
Potential energy change in entire domain

As a whole, “smoothness” of the test function is found to be essential for steep decrease of potential energy.





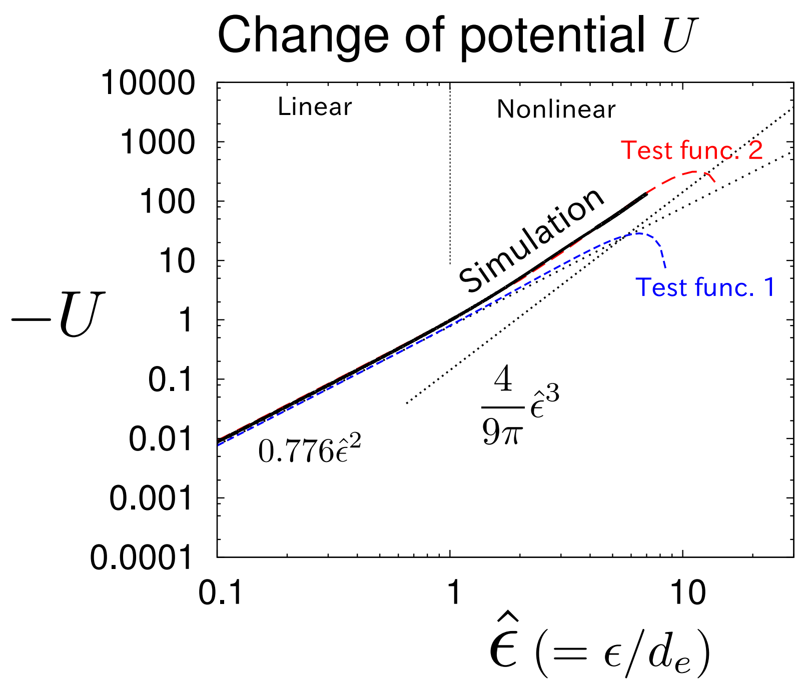
Change of potential U

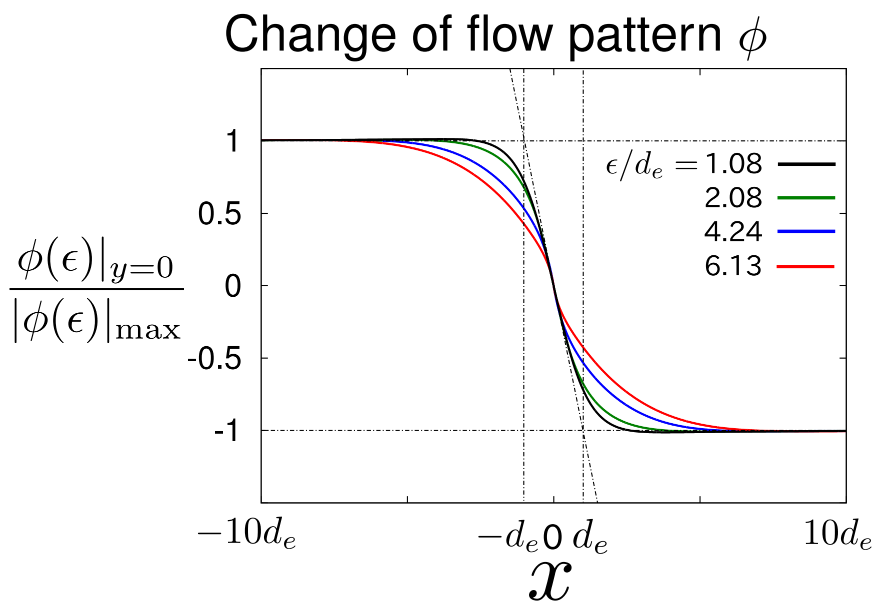


- When the flux ψ_e turns back from the O point side by convection, the potential does not further decrease.
- When the X point elongates and approaches to the “Y-shape”, the potential decreases in cubic power of $\hat{\epsilon}$.

Verification using direct numerical simulation

We calculate the potential energy $U(\hat{\epsilon})$ in the direct numerical simulation.





- Simulation almost agrees with the test function 2 up to $\epsilon < 7d_e$.
- In simulation, flow pattern ϕ tends to smooth gradually in time.

  $\Rightarrow$ The Y-shape seems to be self-organized, searching for the lowest U state.
- The nonlinear acceleration force $F(\hat{\epsilon}) = -U'(\hat{\epsilon}) \sim \hat{\epsilon}^2$ is different from $F(\hat{\epsilon}) \sim \hat{\epsilon}^4$ in Ottaviani & Porceli (1993), but simulation agrees with our scaling.

Summary

- We have performed nonlinear analysis and simulation of magnetic reconnection driven by electron inertia, to clarify its acceleration mechanism.
- By formulating variational principle (Lagrangian) of a two-fluid model, growth of magnetic island can be predicted by finding a test function that minimizes potential energy of the system.
  - In linear phase ($\epsilon \ll d_e$), the exponential growth rate $\epsilon(t) \propto e^{\gamma t}$ is estimated by using a piecewise-linear function that is similar to the eigenfunction. $\boxed{\text{Potential } U(\hat{\epsilon}^2) = -0.776\hat{\epsilon}^2 + O(\hat{\epsilon}^3)}$
  - In nonlinear phase ($d_e < \epsilon \ll L_x$), a smooth test function predicts decrease of potential energy $U \sim -\hat{\epsilon}^3$ which is steeper than the lienar phase.

    $\Rightarrow$ $\boxed{\text{Explosive growth of island } (\epsilon) \text{ during a finite time} \sim \tau_e = (d_e q' \omega_{A0})^{-1}}$

    Although the model is too simple at present, this time scale (for large tokamaks, $\tau_e \sim 100\mu s$) does not contradict the experimental collapse times.
- By taking a form of Y-shape, most part of magnetic energy flowing into the inner layer is converted into kinetic energy.