Hamiltonian structure and current singularities in two-dimensional RMHD

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An essential difference of the plasma theory from the neutral fluid mechanics is in that plasma models may include a varity of "singular perturbations" determining scale hierarchies. Our aim is to explore how a singular perturbation (electron inertia) determines an intrinsic (small) scale that is absent in the scale-invariant MHD system. Here we study the role of Casimir invariants that characterize the "Non-canonical" property of the determing symplectic geometry; we define some different "sub-classes" of canonicalized self-contained mechanics of the reduced MHD. We focus on a scenario of singularity formation[1] where the magnetic stream function contains a hyperbolic saddle.

1. Introduction

A. Reduced magnetohydrodynamics

The reduced MHD equations are known as the slab models of ideal MHD. Here we have introduced the stream function ϕ the vorticity $U = \nabla_\perp^2 \phi$, and the toroidal current $J = \nabla_\perp^2 \psi$. ψ is poroidal flux. Then the two dimensional reduced MHD are given by

$$\dot{U} = [\psi, J] + [U, \phi] \,,$$
$$\dot{\psi} = [\psi, \phi] \,.$$

where dot denotes time derivative and brackets are

$$[A, B] = \partial_x A \partial_y B - \partial_y A \partial_x B \,.$$

B. Non-canonical hamiltonian structure

The plasma fluid models have some conservation laws arising from either symmetries in the Hamiltonian or a "topological defect" (kernel) of the Poisson bracket. The latter constants of motion is called Casimir invariants. A Hamiltonian system that has Casimir invariants is said "non-canonical." P.J. Morrison and R.D. Hazeltine constructed a Poisson bracket for the two-dimensional reduced MHD[2](See Table 1).

TABLE 1. Non-canonical hamiltonian structure of the reduced MHD

Non-Canonical Form	
Poisson bracket	$\{F,G\} = \int \mathrm{d}^2x W_{ij} \left[\frac{\delta F}{\delta \xi_i}, \frac{\delta G}{\delta \xi_j}\right]$ where $W_{ij} = \begin{bmatrix} 0 & \psi \\ \psi & U \end{bmatrix}$
Equations of motion	$\dot{\psi} = \{\psi, H\}$, $\dot{U} = \{U, H\}$, $i = 1, 2$

2. Canonical hamiltonian structure

Morrison has introduced the following new variables.

$$\psi = [Q_1, Q_2] \,,$$
$$U = [Q_1, P_1] + [Q_2, P_2]$$

Then the equations can be cast into the canonical form. However such canonical formulation involves four fields rather than the initial two. We found several different kinds of transformation. The following transformation maintains two fields,

$$\psi = Q^\alpha,$$
$$U = [Q, P] \quad \text{where} \quad \forall \alpha \in \mathbb{Z} \quad (\alpha \neq 0) \,.$$

Which allows us writing down the canonical hamiltonian form(See Table 2). The equations of motion are as follows;

Non-Canonical Form

$$\dot{U} = [\psi, J] + [U, \phi]$$
$$\dot{\psi} = [\psi, \phi]$$

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Canonical Form

$$D_t Q = 0,$$
$$D_t P = \alpha J Q^{\alpha - 1}$$

Where D_t is the convective derivative, $D_t = \partial_t + [\phi, \ ]$. According to the non-canonical hamiltonian formulation there exists conserved Casimir invariants. Our formulation fixes one of these conserved quantities,

$$C = \int \psi U \mathrm{d}^2 x = \int Q^\alpha [Q, P] \, \mathrm{d}^2 x = 0 \,.$$

In this sense, our formulation is to constitute a subclass of the reduced MHD.

TABLE 2. Canonical hamiltonian structure of the reduced MHD

Canonical Form	
Poisson bracket	$\{F,G\} = \sum_i \int \mathrm{d}^2 x \left(\frac{\delta F}{\delta P_i}\frac{\delta G}{\delta Q_i} - \frac{\delta F}{\delta Q_i}\frac{\delta G}{\delta P_i} \right)$
Equations of motion	$\dot{Q}_i = \frac{\delta H}{\delta P_i}$, $\dot{P}_i = -\frac{\delta H}{\delta Q_i}$, $i = 1, 2$

3. Numerical experiments

For the special case of α=2, we solved canonical(or non-canonical) form of equations of motion numerically on a periodic box with a 2/3-dealiased standard pseudo-spectral method. As time advancing routine, we used a fourth-order Adams-Bashforth method. We considered the initial data with

$$Q = \sqrt{2\cos(x) - \cos(2y) + 4} \,,$$
$$P = 0 \,.$$

In Figure 1, we present the numerical solution at times t=0, 0.6 with a resolution of 512^2 Fourier modes. In Figure 2, we present log plots of max toroidal current versus time. We also present log plots of max P field versus time.

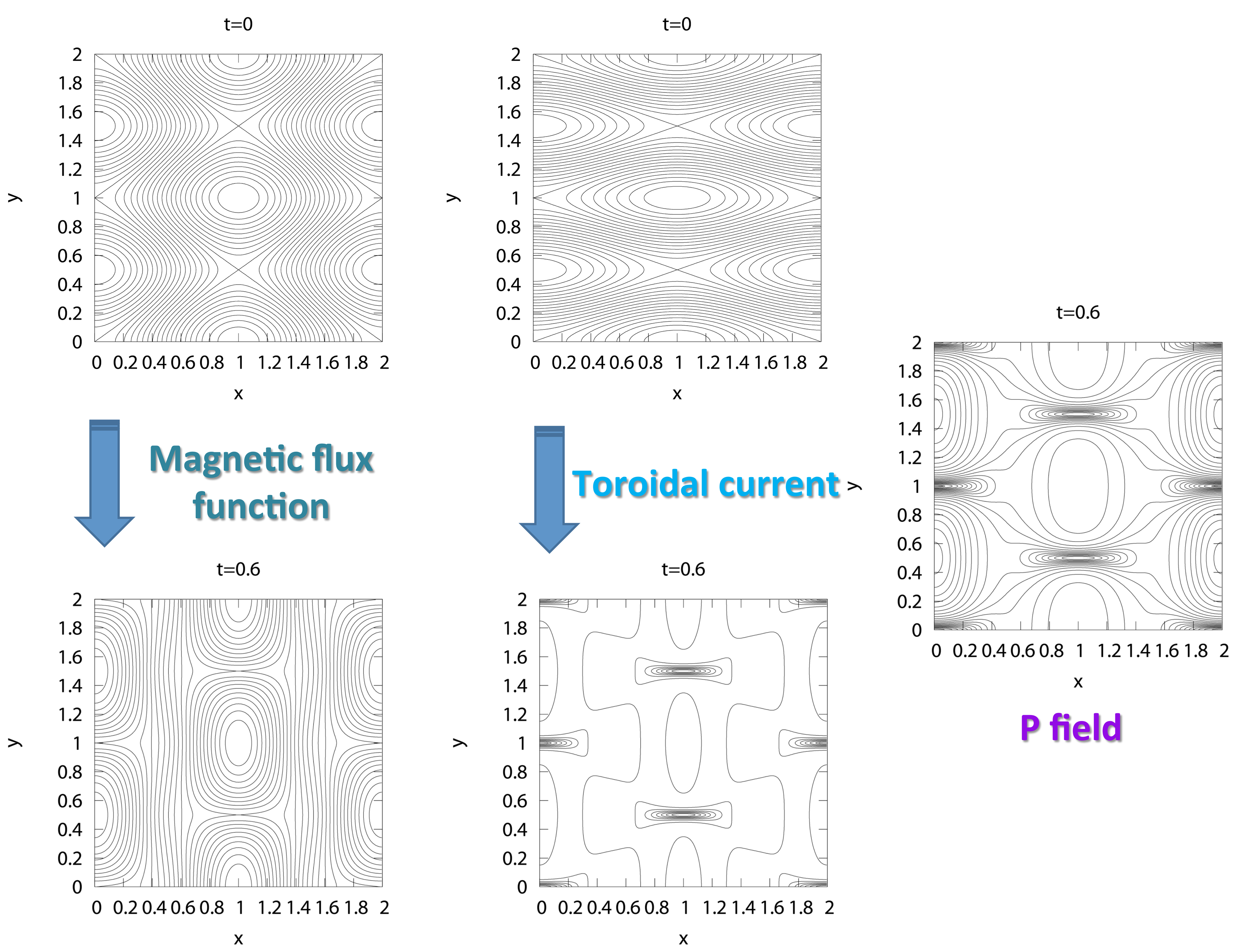

FIGURE 1. Contours of the magnetic flux function, current density and P field

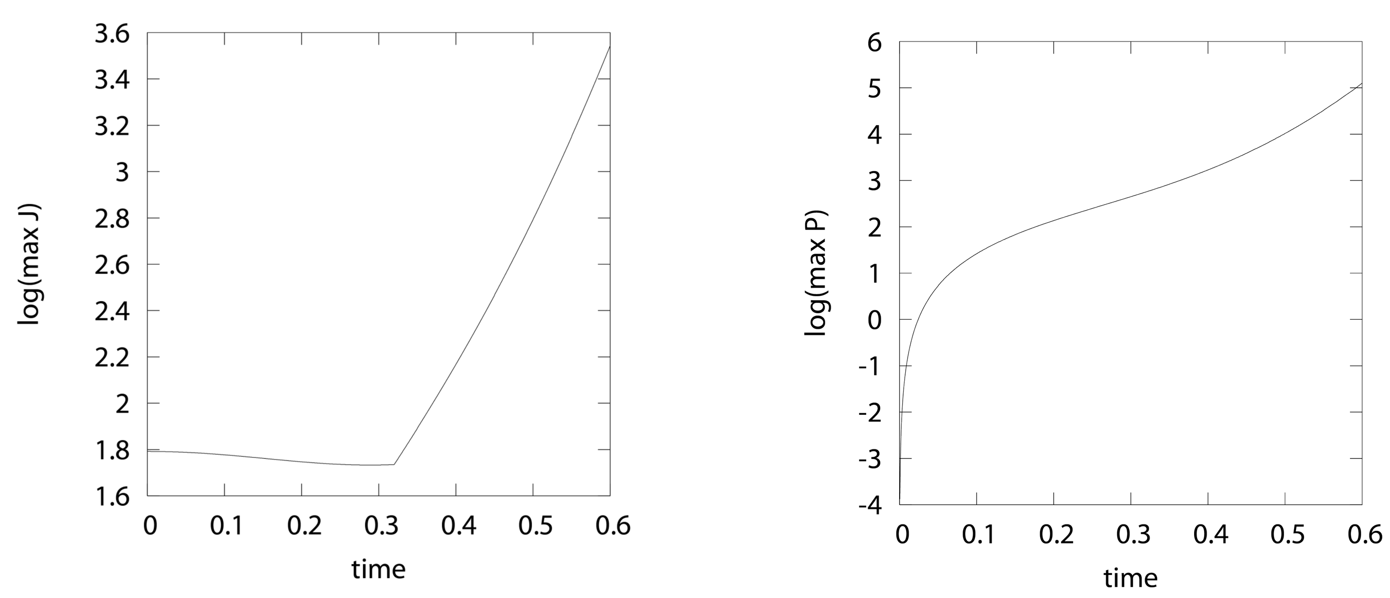

FIGURE 2. log plot of maximum current and maximum P field versus time

4. Numerical Results

The toroidal current becomes extremely large where magnetic field lines closer to each other[3]. The graph turns out to be a clear straight line beyond t〜0.3. It predicts the toroidal current grows like a exponential function of time. On the other hand, the change of P field is quite different from the change of toroidal current. P field is rapidly growing until t〜0.15. Furthermore, it seems to grow faster than the exponential function at later times. As a result, we can predict the growth of P field is faster than the growth of the toroidal current.

5. References

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[2] Morrison, P. J. and Hazeltine, R. D. 1984 *Phys. Fluids* **27**, 886.
[3] D. Cordoba and C. Marliani, Evolution of current sheets and regularity of ideal incompressible magnetic fluids in 2D, *Commun. Pure Appl. Math. 53, 512 (2000).*