On the gyrokinetic model in the long wavelength regime

N. Miyato[1)], B. D. Scott[2)], M. Yagi[1)]

[1)] Japan Atomic Energy Agency

[2)] Max-Planck-Institut für Plasmaphysik

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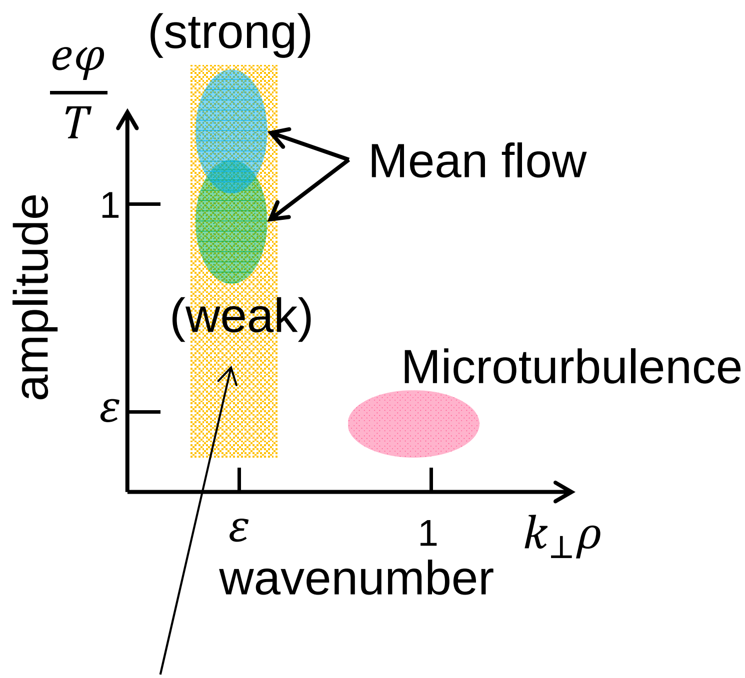
Introduction

- In order to understand formation of transport barriers, we have to consider interactions between microturbulence and background profiles including mean flow self-consistently.
- To this end, global turbulence simulation is needed, and several global codes are developed using the gyrokinetic model.
- But the standard gyrokinetic model is formulated for perturbations with short wavelength and small amplitude (gyrokinetic ordering) [*Hahm PF 1988*].
- Hence, there is no reason that the standard model is still valid in the long wavelength regime.



Is weak mean flow regime covered by the standard gyrokinetic model?

Introduction (cont.)

- It was claimed that the gyrokinetic model would be valid in the long wavelength regime [*Dimits PFB 1992*].
- Recently, on the other hand, it was claimed that gyrokinetic quasi-neutrality equation usually used is not enough to obtain electrostatic potential with long wavelength [*Parra-Catto PPCF 2008*].

- We investigate this issue from a point of view of push-forward representation associated with phase space transformation.
- And show that the 2nd order displacement vector associated with guiding-centre transformation yields additional terms in gyrokinetic quasi-neutrality equation.

Ordering reinterpretation

[*Dimits, Phys. Fluids B 1992*]

Gyrokinetic ordering (short wavelength and small amplitude)

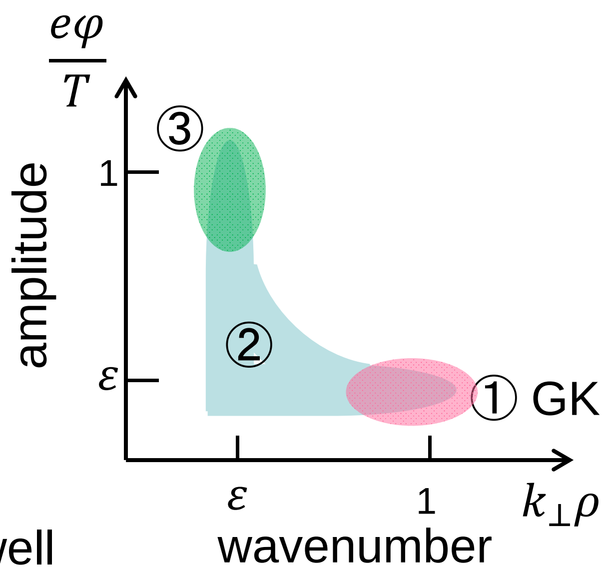
① $k_\perp \rho \sim 1, \frac{e\varphi}{T} \sim \varepsilon_\delta \ll 1$ Microturbulence

Slow flow ordering ($V_E \ll V_{\mathrm{th}}$)

② $k_\perp \rho \frac{e\varphi}{T} \ll 1$

Slow flow ordering is satisfied in long wavelength as well

③ $k_\perp \rho \ll 1, \frac{e\varphi}{T} \sim 1$ Mean flow



Standard gyrokinetic model is also valid in long wavelength regime?

Gyrokinetic quasi-neutrality equation

Gyrokinetic quasi-neutrality equation for singly charged ions (which is the same as push-forward representation of particle density) is

$$n_e = n_i = \bar{N} + n_0(\Gamma_0 - 1)\frac{e\varphi}{T}$$

Polarisation term

where $\bar{N}(\mathbf{r}) = \int d^6\bar{\mathbf{Z}}\delta^3(\bar{\mathbf{X}} - \mathbf{r})\left\langle e^{-\bar{\rho}\cdot\nabla}\right\rangle \bar{F}$ Gyroaveraged gyro-centre density

$$n_0(\mathbf{r}) = \int d^6\bar{\mathbf{Z}}\delta^3(\bar{\mathbf{X}} - \mathbf{r})F_M$$

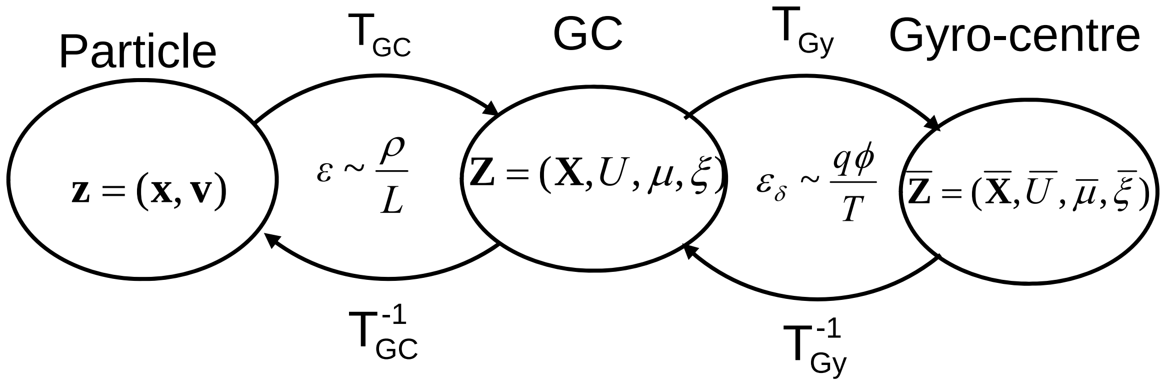
$$\Gamma_0 = \frac{1}{n_0}\int d\bar{U}d\bar{\mu}d\bar{\xi}F_M\left\langle e^{\bar{\rho}\cdot\nabla}\right\rangle^2$$

Observation: Polarisation term is $O(\varepsilon)$ for short wavelength perturbations, but it becomes $O(\varepsilon^2)$ for long wavelength perturbations.

We have to consider other $O(\varepsilon^2)$ terms in long wavelength regime.
For this we have to review the formulation of the gyrokinetic model.

Push-forward representation of particle density

Two-step phase space transformation in standard gyrokinetic formulation

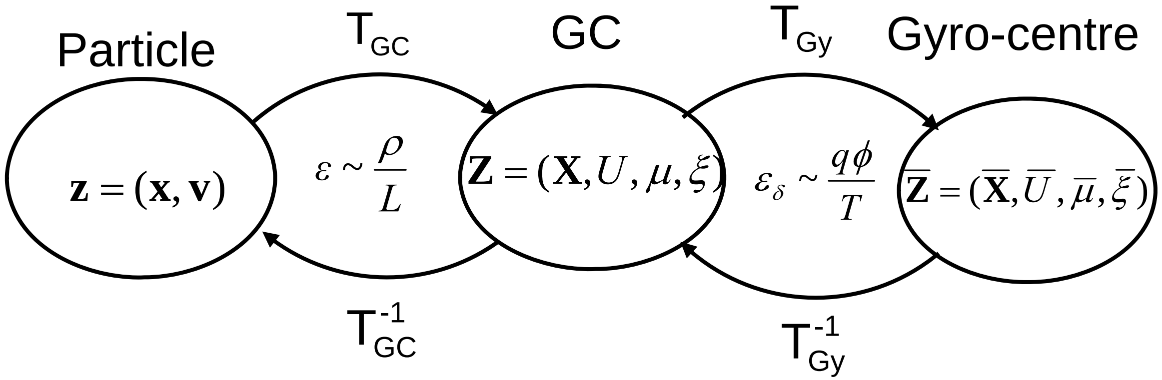


Formal push-forward representation of particle density

$$n(\mathbf{r}) \equiv \int d^3\mathbf{x} d^3\mathbf{v} \, f \delta^3(\mathbf{x} - \mathbf{r}) = \int d^6\bar{\mathbf{Z}} \, \mathcal{J} \mathrm{T}_{\mathrm{Gy}}^{*} \bar{F} \delta^3(\mathrm{T}_{\mathrm{GC}}^{-1}\mathbf{x} - \mathbf{r})$$ Usually used

$$= \int d^6\bar{\mathbf{Z}} \, \mathcal{J} \bar{F} \delta^3(\mathrm{T}_{\mathrm{Gy}}^{-1}\mathrm{T}_{\mathrm{GC}}^{-1}\mathbf{x} - \mathbf{r})$$ Alternative

2nd order pieces of phase space transformation



Particle position in gyro-centre phase space

$$\mathrm{T}_{\mathrm{Gy}}^{-1}\mathrm{T}_{\mathrm{GC}}^{-1}\mathbf{x} = \overline{\mathbf{X}} + \epsilon\overline{\boldsymbol{\rho}} + \epsilon\epsilon_\delta\overline{\boldsymbol{\rho}}_{\mathrm{gy}} + \epsilon^2\overline{\boldsymbol{\rho}}_B + O(\epsilon^3)$$

$$\overline{\boldsymbol{\rho}}_{\mathrm{gy}} = -\{S_1, \overline{\mathbf{X}} + \overline{\boldsymbol{\rho}}\}$$

Here ϵ and ϵ_δ are indexes showing order of each term.

Displacement vector associated with *gyro-centre transformation*. This piece yields polarisation term in quasi-neutrality equation which is $O(\epsilon_\delta)$ for short wavelength, but $O(\epsilon\epsilon_\delta)$ for long wavelength.

Another 2nd order piece associated with *guiding-centre transformation*. This piece is not considered in the standard gyrokinetic model.

2nd order piece associated with guiding-centre transformation

Relation between **x** and **X**

$$\mathbf{x} = \mathbf{X} - \epsilon G_1^{\mathbf{X}} - \epsilon^2 \left(G_2^{\mathbf{X}} - \frac{1}{2}\mathbf{G}_1 \cdot \mathbf{d} G_1^{\mathbf{X}} \right) + \cdots$$

$$\boldsymbol{\rho} = -\, G_1^{\mathbf{X}}, \;\; \boldsymbol{\rho}_B = -\left(G_2^{\mathbf{X}} - \frac{1}{2}\mathbf{G}_1 \cdot \mathbf{d} G_1^{\mathbf{X}} \right)$$

This is related to variation of magnetic field.

where

$$G_2^{\mathbf{X}} = \frac{1}{2}\left(g_1^{\mu}\frac{\partial \boldsymbol{\rho}}{\partial \mu} + g_1^{\xi}\frac{\partial \boldsymbol{\rho}}{\partial \xi} \right) + \boldsymbol{\rho}\left(\frac{U}{\Omega}\hat{b} \cdot \nabla \times \hat{b} \right) - \frac{1}{m}\frac{\partial S_3}{\partial U}\hat{b}$$

$$g_1^{\mu} = -\frac{U\mu}{\Omega}\mathrm{a}_2 \colon \nabla \hat{b} - \frac{mU^2}{B}\frac{W}{\Omega}\hat{c} \cdot \nabla \times \hat{b} - \frac{U\mu}{\Omega}\hat{b} \cdot \nabla \times \hat{b}$$

$$g_1^{\xi} = \frac{mU^2}{2\mu B}\frac{W}{\Omega}\hat{a} \cdot \nabla \times \hat{b} + \frac{U}{\Omega}\mathrm{a}_2 \colon \nabla \hat{b} - \frac{W}{\Omega}\hat{c} \cdot \nabla \log B$$

$$S_3 = \frac{m}{q}\left[-\frac{mU^2}{B}\frac{W}{\Omega}\hat{a} \cdot \nabla \times \hat{b} + \frac{U\mu}{\Omega}\mathrm{a}_2 \colon \nabla \hat{b} - \frac{2}{3}\frac{W}{\Omega}\mu \hat{c} \cdot \nabla \log B \right]$$

$$\hat{a} = \frac{\boldsymbol{\rho}}{\rho}, \; \hat{c} = \hat{a} \times \hat{b}, \mathrm{a}_2 = \frac{\hat{c}\hat{c} - \hat{a}\hat{a}}{4}, \; W = \sqrt{\frac{2\mu B}{m}}$$

General push-forward representation of particle density

Requirement that the charge density should agree with $\sum qn$ yields the general push-forward representation of particle density [*Miyato et al., Plasma Fusion Res. 2011*],

$$n(\mathbf{r}) = -\frac{1}{q}\int d^6\mathbf{Z}\,\mathcal{J}(\mathbf{Z})F(\mathbf{Z})\frac{\delta L_p(\mathbf{Z})}{\delta\varphi(\mathbf{r})}$$

When the symplectic part of L_p does not include the electrostatic potential, the above is rewritten with the GC (GK) Hamiltonian H as

$$n(\mathbf{r}) = \frac{1}{q}\int d^6\mathbf{Z}\,\mathcal{J}(\mathbf{Z})F(\mathbf{Z})\frac{\delta H(\mathbf{Z})}{\delta\varphi(\mathbf{r})}$$

ex.

Hamiltonian $H = q\varphi + \frac{m}{2}U^2 + \mu B - \frac{m}{2}V_E^2 + \frac{\mu}{2\Omega}\nabla_\perp^2\varphi$

yields $n = N + \nabla\cdot\left[\frac{N}{B\Omega}\nabla_\perp\varphi\right] + \nabla_\perp^2\frac{P_\perp}{2q\Omega B}$

General comments on variational derivation

- The variational method only needs the GC (GK) Hamiltonian (or Lagrangian) and doesn't need details of the GC (GK) transformation. (But this time we needed knowledge of details of the phase space transformations for consideration.)
- Therefore, this method is very useful in modern reduced kinetic formulation based on the Lie-transform perturbation analysis.
- Consistency of the reduced system is automatically kept in the variational method.
- Any approximations should be made to the Hamiltonian, and the quasi-neutrality condition (the Poisson equation) should be derived rigorously from the *approximated* Hamiltonian.
- Otherwise, one must be careful about consistency between the Hamiltonian and the quasi-neutrality. If not, conservation properties of the system will be violated.

Modification of Hamiltonian covering long wavelength

When we take into account $\boldsymbol{\rho}_B$, the electrostatic potential may be expanded as

$$\varphi\left(\mathrm{T}_{\mathrm{GC}}^{-1}\mathbf{x}\right) = \varphi(\mathbf{X} + \boldsymbol{\rho} + \boldsymbol{\rho}_B + \cdots) \approx \varphi(\mathbf{X} + \boldsymbol{\rho}) + \boldsymbol{\rho}_B \cdot \nabla\varphi(\mathbf{X} + \boldsymbol{\rho}).$$

Hence, an additional term appears in the gyrokinetic Hamiltonian [*Brizard Diss. 1990*]:

$$\bar{H} = \frac{m}{2}\bar{U}^2 + \bar{\mu}B + \varepsilon_\delta q[\langle\varphi\rangle + \langle\bar{\boldsymbol{\rho}}_B \cdot \nabla\varphi\rangle] - \varepsilon_\delta^2 \frac{q^2}{2B}\frac{\partial}{\partial\mu}(\langle\varphi^2\rangle - \langle\varphi\rangle^2)$$

Since the additional term is important for long wavelength only, we can approximate as $\langle\bar{\boldsymbol{\rho}}_B \cdot \nabla\varphi(\bar{\mathbf{X}} + \bar{\boldsymbol{\rho}})\rangle \approx \langle\bar{\boldsymbol{\rho}}_B\rangle \cdot \nabla\varphi(\bar{\mathbf{X}})$.
Then, the gyrokinetic Hamiltonian becomes

$$\boxed{\begin{aligned}\bar{H}(\bar{\mathbf{X}}, \bar{U}, \bar{\mu}, t) = \frac{m}{2}\bar{U}^2 + \bar{\mu}B(\bar{\mathbf{X}}) + \varepsilon_\delta q[\langle\varphi(\bar{\mathbf{X}} + \bar{\boldsymbol{\rho}})\rangle + \langle\bar{\boldsymbol{\rho}}_B(\bar{\mathbf{X}}, \bar{U}, \bar{\mu})\rangle \cdot \nabla\varphi(\bar{\mathbf{X}})] \\ -\varepsilon_\delta^2 \frac{q^2}{2B(\bar{\mathbf{X}})}\frac{\partial}{\partial\mu}(\langle\varphi(\bar{\mathbf{X}} + \bar{\boldsymbol{\rho}})^2\rangle - \langle\varphi(\bar{\mathbf{X}} + \bar{\boldsymbol{\rho}})\rangle^2)\end{aligned}}$$

Variational derivation

Substituting the modified gyrokinetic Hamiltonian into the general push-forward representation of particle density, we have the representation with the additional terms coming from $\boldsymbol{\rho}_B$:

$$n(\mathbf{r}) = \frac{1}{q}\int d^6\bar{\mathbf{Z}}\mathcal{J}(\bar{\mathbf{Z}})\bar{F}\frac{\delta\bar{H}(\bar{\mathbf{Z}})}{\delta\varphi(\mathbf{r})}$$

$$= \int d^6\bar{\mathbf{Z}}\mathcal{J}\bar{F}\frac{\delta}{\delta\varphi(\mathbf{r})}\left[\varphi(\bar{\mathbf{X}}+\bar{\boldsymbol{\rho}}) - \frac{q}{2B}\frac{\partial}{\partial\mu}\left(\langle\varphi(\bar{\mathbf{X}}+\bar{\boldsymbol{\rho}})^2\rangle - \langle\varphi(\bar{\mathbf{X}}+\bar{\boldsymbol{\rho}})\rangle^2\right) + \langle\bar{\boldsymbol{\rho}}_B\rangle \cdot \boldsymbol{\nabla}\varphi(\bar{\mathbf{X}})\right]$$

$$= \int d^6\bar{\mathbf{Z}}\delta^3(\bar{\mathbf{X}}-\mathbf{r})\left[e^{-\boldsymbol{\rho}\cdot\nabla}\left\{\bar{F}\mathcal{J} + \frac{q}{B}\frac{\partial\bar{F}\mathcal{J}}{\partial\mu}\left(e^{\boldsymbol{\rho}\cdot\nabla}\varphi(\bar{\mathbf{X}}) - \langle e^{\boldsymbol{\rho}\cdot\nabla}\rangle\varphi(\bar{\mathbf{X}})\right)\right\} - \boldsymbol{\nabla}\cdot\mathcal{J}\bar{F}\langle\bar{\boldsymbol{\rho}}_B\rangle\right]$$

We need an explicit form of $\boldsymbol{\rho}_B$.

Explicit representation of ρ_B

Explicit representation of ρ_B is found in [*Littlejohn Phys. Fluids 1981*]

$$\boldsymbol{\rho}_B = -\frac{1}{\Omega^2}\Bigg[U^2\hat{b}\cdot\nabla\hat{b} + UW\left\{\frac{1}{2}(\hat{b}\cdot\nabla\times\hat{b} - \mathrm{a}_1:\nabla\hat{b})\hat{a} + 2(\hat{b}\cdot\nabla\hat{b}\cdot\hat{c})\hat{b} + (\mathrm{a}_2:\nabla\hat{b})\hat{c}\right\} + \frac{\mu B}{m}\left\{\left(\frac{1}{2}\nabla\cdot\hat{b} - \mathrm{a}_2:\nabla\hat{b}\right)\hat{b} + \frac{3}{2}\nabla_\perp\log B + 2\mathrm{a}_2\cdot\nabla\log B\right\}\Bigg]$$

(Littlejohn gives more general $\boldsymbol{\rho}_B$ including equilibrium electric field.)

Cf. [*Brizard Diss.1990*]

$$\boldsymbol{\rho}_B = -\frac{1}{\Omega^2}\left(U^2\hat{b}\cdot\nabla\hat{b} + 2\frac{\mu B}{m}\nabla_\perp\log B\right) - \hat{b}\left[\frac{\mu B}{4m\Omega^2}(\nabla\cdot\hat{b}) + 2\frac{UW}{\Omega^2}(\hat{b}\cdot\nabla\hat{b}\cdot\hat{c})\right] - \frac{U}{2\Omega}\boldsymbol{\rho}\cdot\left[\mathbf{I}(\hat{b}\cdot\nabla\times\hat{b}) + \frac{1}{2}(\hat{b}\times\nabla\hat{b} - \nabla\hat{b}\times\hat{b})\right]$$

Additional terms in quasi-neutrality

Now we need only gyroaverage of $\boldsymbol{\rho}_B$,

$$\langle \overline{\boldsymbol{\rho}}_B \rangle = -\left[\frac{\mu B}{m\Omega^2}\frac{1}{2}(\nabla \cdot \hat{b})\hat{b} + \frac{U^2}{\Omega^2}\hat{b}\cdot\nabla\hat{b} + \frac{3}{2}\frac{\mu B}{m\Omega^2}\nabla_\perp \log B\right].$$

This is a little different from recent result by Brizard and Tronko,

$$\langle \overline{\boldsymbol{\rho}}_B \rangle = -\left[\frac{U^2}{\Omega^2}\hat{b}\cdot\nabla\hat{b} + \frac{\mu B}{m\Omega^2}\nabla_\perp \log B\right].$$ [*Brizard-Tronko, PoP 2011*]

It does not agree with gyroaverage of Brizard's $\boldsymbol{\rho}_B$ in the previous slide.

Then, the quasi-neutrality becomes

$$n(\mathbf{r}) = \underset{\substack{\uparrow \\ \text{Conventional part}}}{(\cdots)} + \nabla\cdot\left[\frac{P_\perp}{m\Omega^2}\frac{1}{2}(\nabla\cdot\hat{b})\hat{b} + \frac{P_\parallel + mNV_\parallel^2}{m\Omega^2}\hat{b}\cdot\nabla\hat{b} + \frac{3}{2}\frac{P_\perp}{m\Omega^2}\nabla_\perp \log B\right].$$

(N, $V_\parallel$, $P_\perp$, $P_\parallel$ are gyro-fluid moments.)

Note: we have similar terms in large flow case [*Miyato et al JPSJ 2009*].

Summary

- Standard gyrokinetic model is not always valid in the long wavelength regime. In particular polarisation term in the quasi-neutrality goes to higher order.
- We have to consider the high order displacement vector associated with GC transformation as well as that associated with gyro-centre transformation yielding polarisation term.
- Considering the high order displacement vector, we obtain additional terms in gyrokinetic quasi-neutrality which come from nonuniformity of magnetic field.
- Our result is slightly different from the recent result of Brizard-Tronko (and our result in large flow case).
  Cause of the discrepancy is unknown yet.
  [There is no detail in Brizard-Tronko which just cites Kaufman's paper which also just cites Wimmel's paper.]

Variational method

An action functional I with a Lagrangian for the Vlasov-Poisson system [*Sugama, Phys. Plasmas 2000*],

$$
\begin{aligned}
I &= \int_{t_1}^{t_2} L dt, \\
L &= \sum \int d^6\mathbf{Z}' \mathcal{J}(\mathbf{Z}') F(\mathbf{Z}', t) L_p[\mathbf{Z}_p(\mathbf{Z}', t'; t), \dot{\mathbf{Z}}_p(\mathbf{Z}', t'; t), t] - \int d^3\mathbf{x} \frac{1}{4\mu_0} \mathsf{F} : \mathsf{F},
\end{aligned}
$$

where $L_p = q\mathbf{A}^* \cdot \dot{\mathbf{X}} + \frac{m}{q}\dot{\xi} - H$, $\mathbf{Z}_p(\mathbf{Z}', t'; t)$ is the GC (or gyro-center) coordinates of the particle at time t with the initial condition $\mathbf{Z}_p(\mathbf{Z}', t'; t') = \mathbf{Z}'$, Σ is a sum over species and $\mathbf{F}_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, A_\mu = (-\varphi/c, \mathbf{A})$, $\partial_\mu = (c^{-1}\partial_t, \boldsymbol{\nabla})$. Here we take $\eta_{ij} = \mathrm{diag}(-1, +1, +1, +1)$ as a metric tensor of Minkowski spacetime.

From $\delta I/\delta\varphi = 0$, we obtain the reduced Poisson equation,

$$
\epsilon_0 \boldsymbol{\nabla}^2 \varphi = \underbrace{\sum \int d^6\mathbf{Z}\, \mathcal{J}(\mathbf{Z}) F(\mathbf{Z}) \frac{\delta L_p(\mathbf{Z})}{\delta\varphi(\mathbf{r})}}_{\text{Charge density is expressed in terms of GC variables.}}
$$

Perturbative expansion of formal exact representation

Explicit push-forward representation of particle density is also derived by a perturbative expansion of formal exact representation.

Two exact representations are possible in the standard gyrokinetic model.

$$n(\mathbf{r}) = \int d^6\bar{\mathbf{Z}}\,\mathcal{J}\bar{F}\delta^3(\mathrm{T}_{\mathrm{Gy}}^{-1}\mathrm{T}_{\mathrm{GC}}^{-1}\mathbf{x} - \mathbf{r})$$

$$= \int d^6\bar{\mathbf{Z}}\,\mathcal{J}\mathrm{T}_{\mathrm{Gy}}^{*}\bar{F}\delta^3(\mathrm{T}_{\mathrm{GC}}^{-1}\mathbf{x} - \mathbf{r})$$

Usually used representation

$$\mathrm{T}_{\mathrm{Gy}}^{-1}\mathrm{T}_{\mathrm{GC}}^{-1}\mathbf{x} = \bar{\mathbf{X}} + \bar{\boldsymbol{\rho}} + \bar{\boldsymbol{\rho}}_{\mathrm{gy}} + \bar{\boldsymbol{\rho}}_{B} + \cdots$$

$$\mathrm{T}_{\mathrm{GC}}^{-1}\mathbf{x} = \mathbf{X} + \boldsymbol{\rho} + \boldsymbol{\rho}_{B} + \cdots$$

$$\mathrm{T}_{\mathrm{Gy}}^{*}\bar{F} = \bar{F} + \{S_1, \bar{F}\} + \cdots$$

Expanding the delta function in powers of $\boldsymbol{\rho}_B$ and integrating by parts, we obtain the same additional term derived by the variational method.

Variational derivation

Substituting the modified gyrokinetic Hamiltonian into the general push-forward representation of particle density, we have the representation with the additional terms coming from $\boldsymbol{\rho}_B$:

$$n(\mathbf{r}) = \frac{1}{q}\int d^6\bar{\mathbf{Z}}\mathcal{J}(\bar{\mathbf{Z}})\bar{F}\frac{\delta \bar{H}(\bar{\mathbf{Z}})}{\delta\varphi(\mathbf{r})}$$

$$= \int d^6\bar{\mathbf{Z}}\mathcal{J}\bar{F}\frac{\delta}{\delta\varphi(\mathbf{r})}\left[\varphi(\bar{\mathbf{X}}+\bar{\boldsymbol{\rho}}) - \frac{q}{2B}\frac{\partial}{\partial\mu}(\langle\varphi(\bar{\mathbf{X}}+\bar{\boldsymbol{\rho}})^2\rangle - \langle\varphi(\bar{\mathbf{X}}+\bar{\boldsymbol{\rho}})\rangle^2) + \langle\bar{\boldsymbol{\rho}}_B\rangle \cdot \nabla\varphi(\bar{\mathbf{X}})\right]$$

$$= \int d^6\bar{\mathbf{Z}}\delta^3(\bar{\mathbf{X}}-\mathbf{r})\left[e^{-\boldsymbol{\rho}\cdot\nabla}\left\{\bar{F}\mathcal{J} + \frac{q}{B}\frac{\partial\bar{F}\mathcal{J}}{\partial\mu}(e^{\boldsymbol{\rho}\cdot\nabla}\varphi(\bar{\mathbf{X}}) - \langle e^{\boldsymbol{\rho}\cdot\nabla}\rangle\varphi(\bar{\mathbf{X}}))\right\} - \nabla\cdot\mathcal{J}\bar{F}\langle\bar{\boldsymbol{\rho}}_B\rangle\right]$$

$$= (\cdots) + \nabla\cdot\left[\frac{P_\perp}{m\Omega^2}\frac{1}{2}(\nabla\cdot\hat{b})\hat{b} + \frac{P_\parallel + mNV_\parallel^2}{m\Omega^2}\hat{b}\cdot\nabla\hat{b} + \frac{3}{2}\frac{P_\perp}{m\Omega^2}\nabla_\perp \log B\right].$$

Conventional part

Large flow case

Second order Hamiltonian in large flow case is given by [*Miyato JPSJ 2009*]

$$H_2 = \frac{m\mu}{2q}\hat{b}\cdot\nabla\times\mathbf{V}_E - \frac{mU^2}{B\Omega}\nabla\varphi\cdot\left(\hat{b}\cdot\nabla\right)\hat{b} - \frac{7}{6}\frac{\mu}{\Omega}\nabla_\perp\varphi\cdot\nabla\log B$$

where nonlinear terms of φ are dropped.

This gives

$$\int d^6\mathbf{Z}\,\mathcal{J}F\,\frac{\delta H_2(\mathbf{Z})}{\delta\varphi(\mathbf{r})}$$

$$= \int d^6\mathbf{Z}\,\delta^3(\mathbf{X}-\mathbf{r})\boldsymbol{\nabla}\cdot\left[\frac{\boldsymbol{\nabla}_\perp F\mathcal{J}\mu}{2\Omega} - \frac{F\mathcal{J}\mu}{2\Omega}\hat{b}\cdot\boldsymbol{\nabla}\hat{b} + \frac{mU^2F\mathcal{J}}{B\Omega}\hat{b}\cdot\boldsymbol{\nabla}\hat{b} + \frac{7}{6}\frac{\mu F\mathcal{J}}{\Omega}\boldsymbol{\nabla}_\perp\log B\right]$$

$$= \int d^6\mathbf{Z}\,\delta^3(\mathbf{X}-\mathbf{r})\boldsymbol{\nabla}\cdot\left[\boldsymbol{\nabla}_\perp\frac{F\mathcal{J}\mu}{2\Omega} + \frac{\mu F\mathcal{J}}{2\Omega}\boldsymbol{\nabla}_\perp\log B - \frac{F\mathcal{J}\mu}{2\Omega}\hat{b}\cdot\boldsymbol{\nabla}\hat{b} + \frac{mU^2F\mathcal{J}}{B\Omega}\hat{b}\cdot\boldsymbol{\nabla}\hat{b} + \frac{7}{6}\frac{\mu F\mathcal{J}}{\Omega}\boldsymbol{\nabla}_\perp\log B\right]$$

Summary of equations

The gyrokinetic Vlasov equation

$$\frac{\partial \bar{F}}{\partial t} + \{\bar{F}, \bar{H}\} = 0 \qquad \langle\{S_1, \tilde{\varphi}\}\rangle \approx \frac{q}{B}\frac{\partial}{\partial \mu}\langle\tilde{\varphi}^2\rangle = \frac{q}{B}\frac{\partial}{\partial \mu}(\langle\varphi^2\rangle - \langle\varphi\rangle^2).$$

with

$$\bar{H} = \frac{m}{2}\bar{U}^2 + \bar{\mu}B + q\langle\varphi\rangle - \frac{q}{2}\langle\{S_1, \tilde{\varphi}\}\rangle + q\langle\bar{\boldsymbol{\rho}}_B \cdot \boldsymbol{\nabla}\varphi\rangle.$$

The gyrokinetic quasi-neutrality equation

$$n(\mathbf{r}) = \int d^6\bar{\mathbf{Z}} J(\bar{\mathbf{Z}})\delta^3(\bar{\mathbf{X}} + \bar{\boldsymbol{\rho}} - \mathbf{r})[\bar{F} + \{S_1, \bar{F}\}] - \int d^6\bar{\mathbf{Z}}\delta^3(\bar{\mathbf{X}} - \mathbf{r})\boldsymbol{\nabla}\cdot J(\bar{\mathbf{Z}})\bar{F}\langle\bar{\boldsymbol{\rho}}_B\rangle$$

where

$$\langle\bar{\boldsymbol{\rho}}_B\rangle = -\left[\frac{\mu B}{m\Omega^2}\frac{1}{2}\left(\boldsymbol{\nabla}\cdot\hat{b}\right)\hat{b} + \frac{U^2}{\Omega^2}\hat{b}\cdot\boldsymbol{\nabla}\hat{b} + \frac{3}{2}\frac{\mu B}{m\Omega^2}\boldsymbol{\nabla}_\perp \log B\right].$$

Note: Usually variation of $\bar{\rho}$ is neglected in the quasi-neutrality. But it is not negligible in long wavelength regime, since it is comparable to $\bar{\rho}_B$.