Variational approach to spectral stability of flowing plasmas

Makoto Hirota[1], Philip J. Morrison[2], Yuji Hattori[1]

[1] Institute of Fluid Science, Tohoku University

[2] Institute for Fusion Studies, The University of Texas at Austin

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Outline

Linear stability condition for flowing plasma is difficult to derive theoretically, even for ideal plasmas.

- Non-selfadjoint eigenvalue problem ($\Leftrightarrow$ shear flow)
- Singular differential equation ($\Leftrightarrow$ continuous spectrum)
- Variational methods give only sufficient conditions for stability
  (e.g. Energy principle)

$\Rightarrow$ It is hard to prove instability for a given equilibrium flow (unless the rare analytical solution exists).

In this work, we attempt to improve the variational method so as to give the necessary and sufficient condition for stability.

- Rayleigh equation (i.e., stability of parallel shear flow)
- Vlasov-Poisson equation
- Ideal MHD equation

Energy principle (Rayleigh-Ritz variational method)

Linear perturbations of ideal fluid and plasma are governed by "Newton's 2nd law of motion" in terms of displacement field $\boldsymbol{\xi}$. [Low 1958, Frieman & Rotenberg 1960]

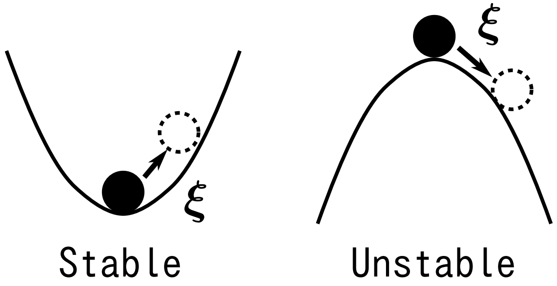
For ideal MHD, $$\rho\left(\frac{\partial}{\partial t}+\boldsymbol{U}\cdot\nabla\right)^2\boldsymbol{\xi}=\mathcal{F}\boldsymbol{\xi}$$

In the absence of flow ($\boldsymbol{U}=0$), the necessary and sufficient condition for spectral stability is that the potential energy $\delta^2 W=-\int\boldsymbol{\xi}\cdot\mathcal{F}\boldsymbol{\xi}d^3x$ is positive definite.

[Bernstein $et\ al.$ 1958]

Energy principle : $$(\text{Growth rate})^2=\max_{\boldsymbol{\xi}}\frac{\int\boldsymbol{\xi}\cdot\mathcal{F}\boldsymbol{\xi}d^3x}{\int\rho|\boldsymbol{\xi}|^2d^3x}$$

- $\mathcal{F}$ is selfadjoint with respect to the norm $\int\rho|\boldsymbol{\xi}|^2d^3x$.
- We can prove instability without solving the equation of motion.
- Instability mechanism is intuitive.



Difficulty of flow stability

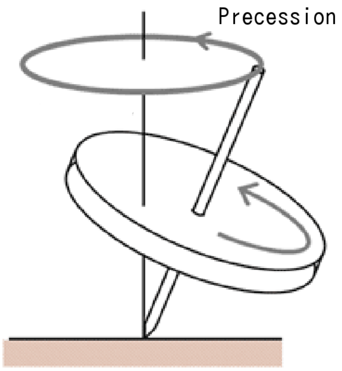
However, the Energy principle is not valid in the presence of flow $\boldsymbol{U}$.

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} + \underbrace{2\rho \boldsymbol{U} \cdot \nabla \frac{\partial \boldsymbol{\xi}}{\partial t}}_{\text{Coriolis force}} = \mathcal{F}\boldsymbol{\xi} - \underbrace{\rho \boldsymbol{U} \cdot \nabla (\boldsymbol{U} \cdot \nabla \boldsymbol{\xi})}_{\text{Centrifugal force}}$$

- The potential energy is positive definite $\Rightarrow$ Stable
  (only sufficient condition)
- Owing to the gyroscopic term, the equilibrium can be stable even when the potential energy is negative.

Analogy: Precession of spinning top

This neutrally stable mode has negative perturbation energy (which is destabilized by dissipation).

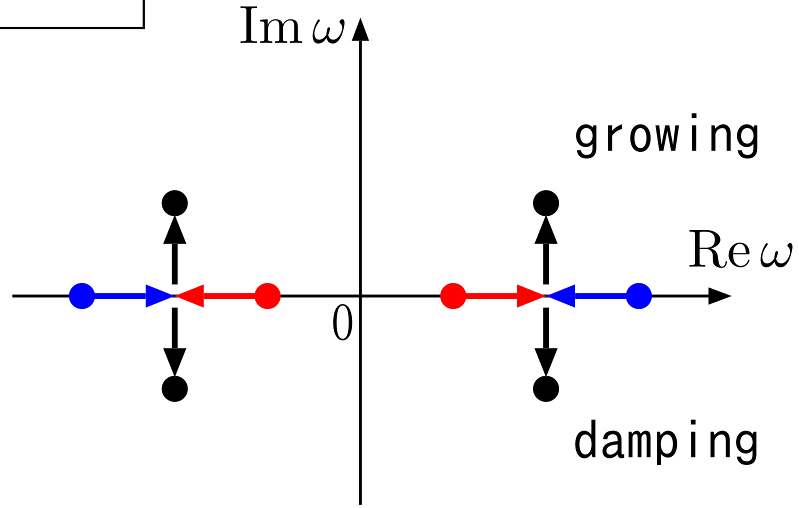


$\Rightarrow$ Variational method cannot obtain the necessary and sufficient condition!

Difficulty of flow stability (continued)

Krein's theory (1950): Instability of Hamiltonian system is cause by resonace between positive energy mode and negative energy mode.

Complex frequency plane



(Hamiltonian Hopf bifurcation)

- However, it is difficult to predict this instability unless the eigenvalue problem is actually solved.
- Krein's theory is not established for continuous spectrum.

☞ We first consider the Rayleigh equation as the simplest problem of flow stability.

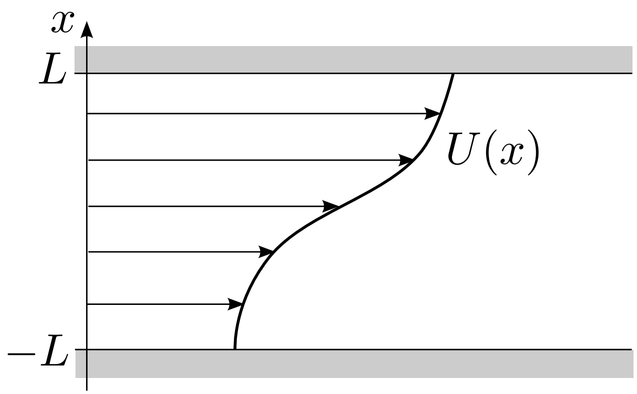
Rayleigh equation ~ Stability of inviscid parallel shear flow ~

Basic flow $\boldsymbol{U} = U(x)\boldsymbol{e}_y$, Distrubance $\tilde{\boldsymbol{u}} = \nabla[\phi(x)e^{-i\omega t+iky} + \mathrm{c.c.}] \times \boldsymbol{e}_z$, ($\omega \in \mathbb{C}$, $k \in \mathbb{R}$)

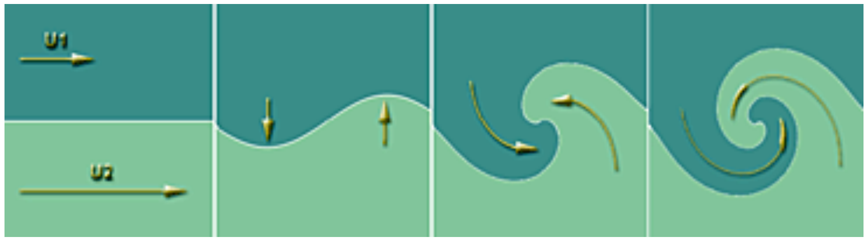
$$(c-U)(\phi'' - k^2\phi) + U''\phi = 0, \qquad \phi(-L) = \phi(L) = 0$$

If there exists an eigenvalue $c = \omega/k$ with $\operatorname{Im} c > 0$, the flow is spectrally unstable.

- The most classical hydrodynamic stability problem
- Non-selfadjoint eigenvalue problem with singularity (continuous spectrum)
- Stability boundary is sensitive to the velocity profile $U(x)$ and still nontrivial.

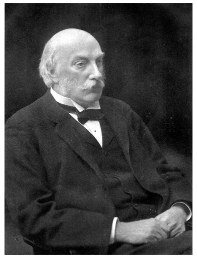


Kelvin-Helmholtz instability



History

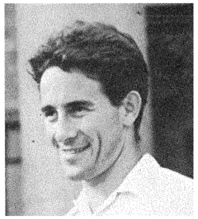
- 1880 Rayleigh
  No inflection point ($U'' \neq 0$) $\Rightarrow$ Stable



- 1950 Fjørtoft
  One inflection point x_I and $U''(U-U_I) > 0$ where $U_I = U(x_I)$ $\Rightarrow$ Stable

- 1964 Rosenbluth & Simon (Nyquist method)
  In the limit $k \to 0$,
  $$\left.\frac{1}{U'(U-U_I)}\right|_{-L}^{L} + \int_{-L}^{L} \frac{U''}{U'^2(U-U_I)} dx > 0 \Leftrightarrow \text{Stable}$$

- 1969 Arnold (variational method)
  $\delta^2 E$ is poitive or negative definite $\Rightarrow$ Stable



- 1999 Balmforth & Morrison (Nyquist method) $\star$
- 2003, 2005 Lin (justification of Tollmien's heuristic method) $\star$

$\star$ These methods need to solve the Rayleigh equation in some way, and the necessary and sufficient condition is still ambiguous when there are multiple inflection points.

☞ We will improve Arnold's variational method and present the necessary and sufficient stability condition for a class of shear flows.

Arnold's variational method

(Arnold 1969) The shear flow U is stable if the second variation of the energy,

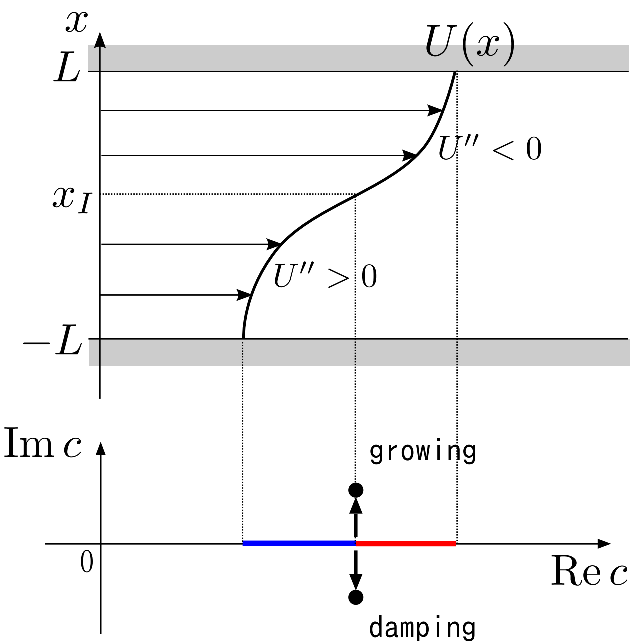
$$\delta^2 E = \int_{-L}^{L} (\boldsymbol{U} \cdot \delta^2 \boldsymbol{u} + |\delta \boldsymbol{u}|^2) dx = \int_{-L}^{L} \overline{\xi} U''[U\xi - \Delta^{-1}(U''\xi)] dx,$$

is either positive or negative definite, where ξ is the fluid displacement and $\Delta\phi := \phi'' - k^2\phi$.

Remarks

- Rayleigh equation has a continuous spectrum $c = \omega/k \in \{U(x) \in \mathbb{R}\ ;\ x \in [-L, L]\}$.
  (Case 1960)
- Sign of the energy of continuous spectrum
  $=$ Sign of UU''
  (Balmforth & Morrison 2002, Hirota & Fukumoto 2008)
- Kelvin-Helmholtz instability occurs at a contact point between positive- and negative-energy continuous spectra.

$\cdots$ Analogous to Krein's theory

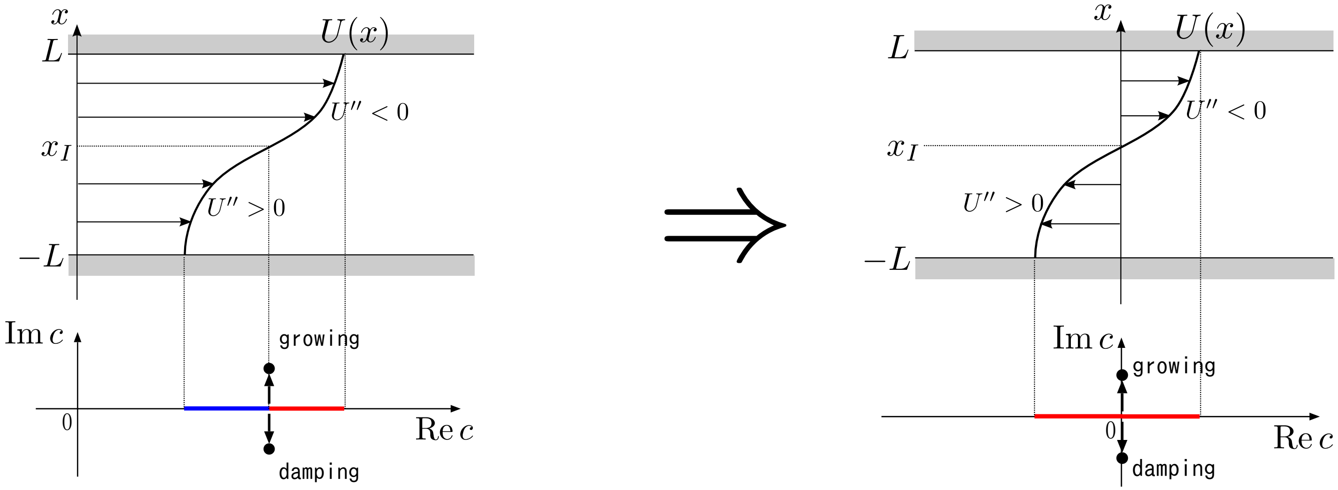


Basic idea of our approach

Suppose that $U(x)$ has only one inflection point $x = x_I$.
In the inertial frame moving at the velocity $U_I = U(x_I)$, the energy $\delta^2 E$ becomes

$$\delta^2 E_I = \int_{-L}^{L} \bar{\xi} U'' [(U - U_I)\xi - \Delta^{-1}(U''\xi)] dx.$$

- If $U''(U - U_I) \geq 0$ for all x, the flow is stable (Fjørtoft 1950).
- If $U''(U - U_I) \leq 0$ for all x,



In this frame, the energy of the continuous spectrum is all negative.

$\Rightarrow$ The flow must be unstable if and only if $\delta^2 E_I > 0$ for some ξ.

Necessary and sufficient stability condition

Assumption:

1) $U(x)$ is an analytic, bounded and strictly monotonic function on $[-L, L]$.
2) if $U''(x_I) = 0$ at $x = x_I$, then $U'''(x_I) \neq 0$.

Theorem:

Denote the inflection points of $U(x)$ by x_{In}, $n = 1, 2, \ldots, N$, and define $U_{In} = U(x_{In})$. The shear flow is spectrally stable if and only if the quadratic form,

$$Q = \nu \int_{-L}^{L} \xi \prod_{n=1}^{N} \left[(U - U_{In}) - U'' \Delta^{-1} \right] U'' \xi dx,$$

is not positive for all $\xi \in \mathrm{L}^2$, where either $\nu = 1$ or $\nu = -1$ is chosen such that

$$\nu U'' \prod_{n=1}^{N} (U - U_{In}) \leq 0 \quad \text{for all } x.$$

Namely,

$$\max_{\xi} \frac{Q}{\|\xi\|^2} > 0 \quad \Leftrightarrow \quad \text{Unstable}$$

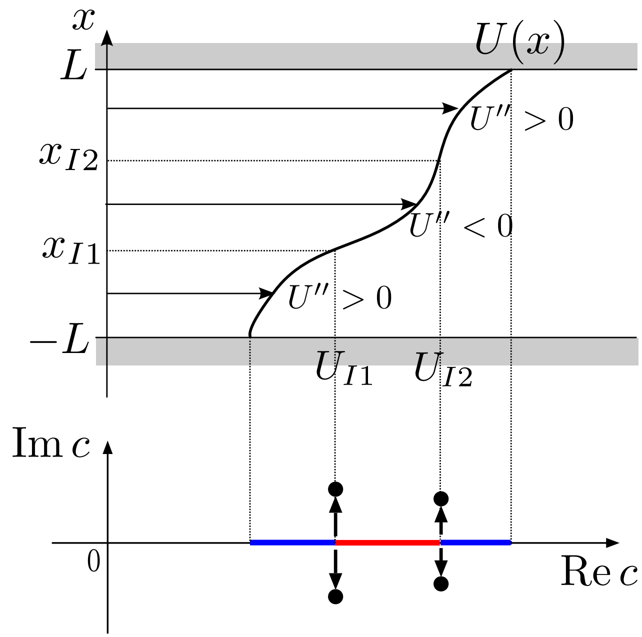
- $Q \leq 0$ for the continuous spectrum

  $Q > 0$ for the discrete spectra

- Number of the positive signature of Q is equal to number of the unstable modes.

- By extending the functional space L^2 of ξ to other Hilbert spaces, one can technically remove the continuous spectrum.

- Unlike Rayleigh-Ritz method, the value $\max Q/\|\xi\|^2$ is not quantitatively related to the growth rate.



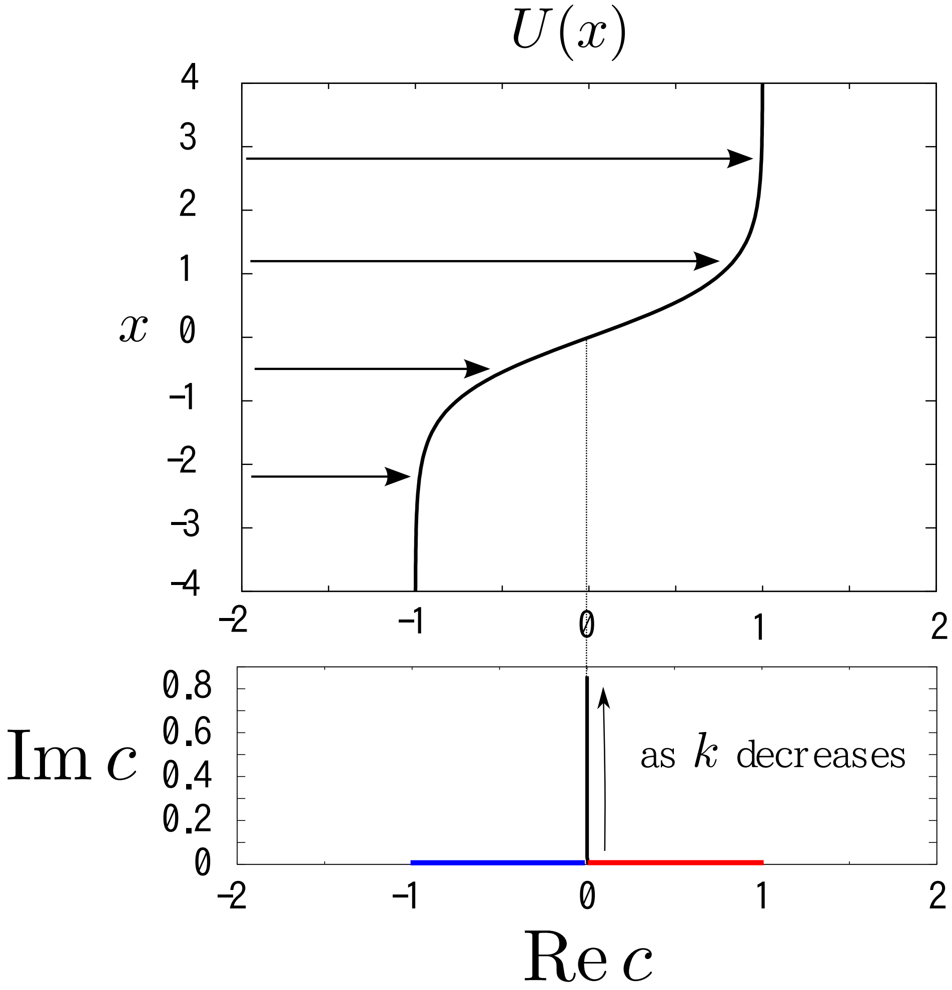
Numerical verification

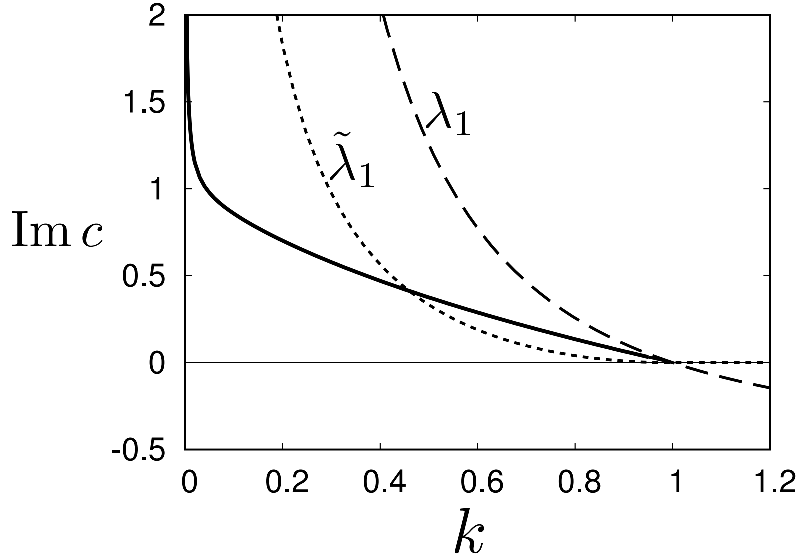
For a given velocity profile $U(x)$, we compare the results of two different numerical codes;

Rayleigh equation	Variational criterion $\max Q/\|\xi\|^2$
non-selfadjoint	selfadjoint
$c_1, c_2, \cdots \in \mathbb{C}$	$\lambda_1 > \lambda_2 > \cdots \in \mathbb{R}$
$\phi(x) \in \mathbb{C}$	$\xi(x) \in \mathbb{R}$
$\mathrm{Im}\, c_j > 0 \Leftrightarrow$ Unstable	$\lambda_1 > 0 \Leftrightarrow$ Unstable
singular as $\mathrm{Im}\, c \to +0$	non-singular around $\lambda = 0$

Example 1

$$U(x) = \tanh(x), \quad x \in [-\infty, \infty]$$



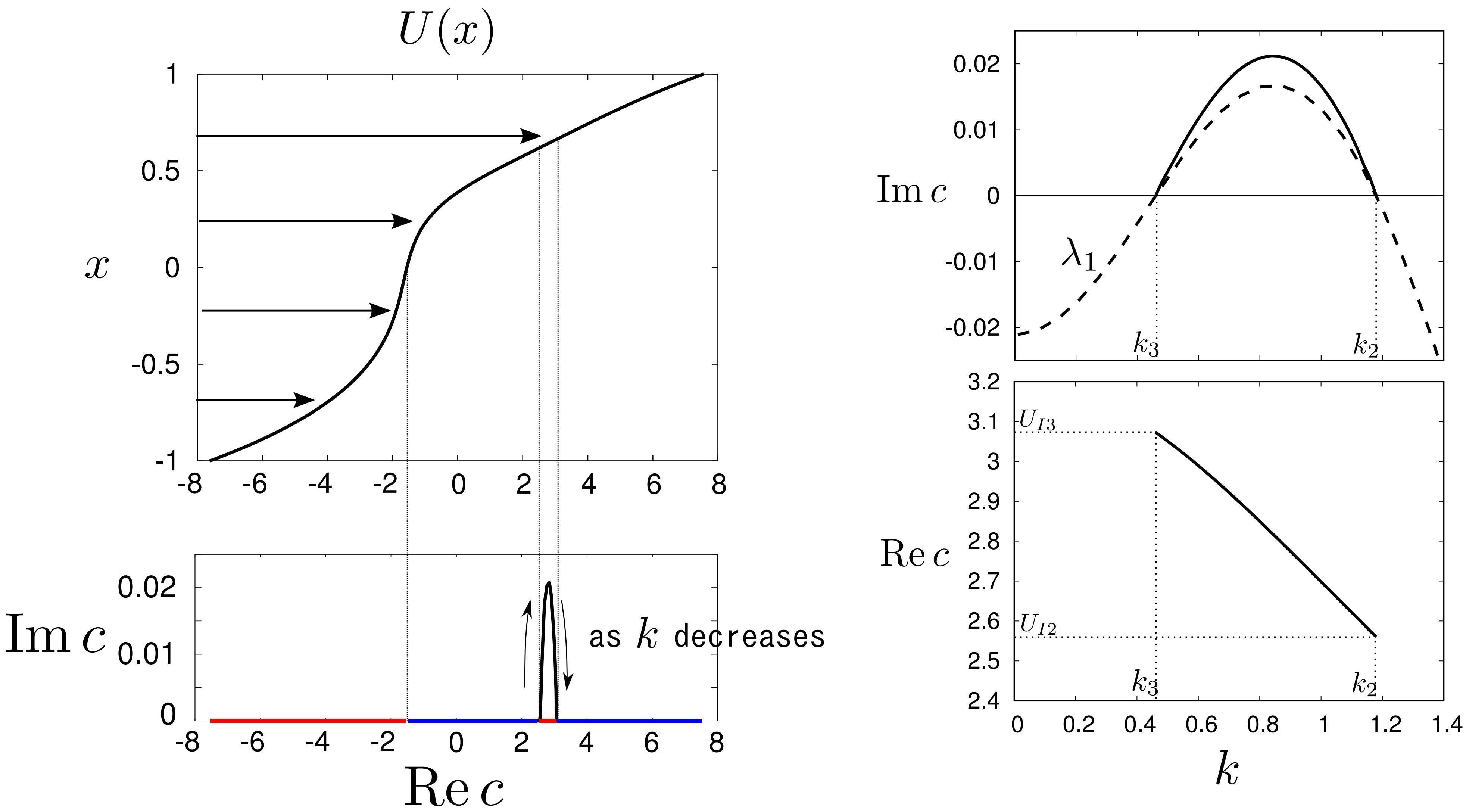


$$\tilde{\lambda}_1 = \max Q / \int |\xi|^2 dx$$

$$\lambda_1 = \max Q / \int |U''\xi|^2 dx$$

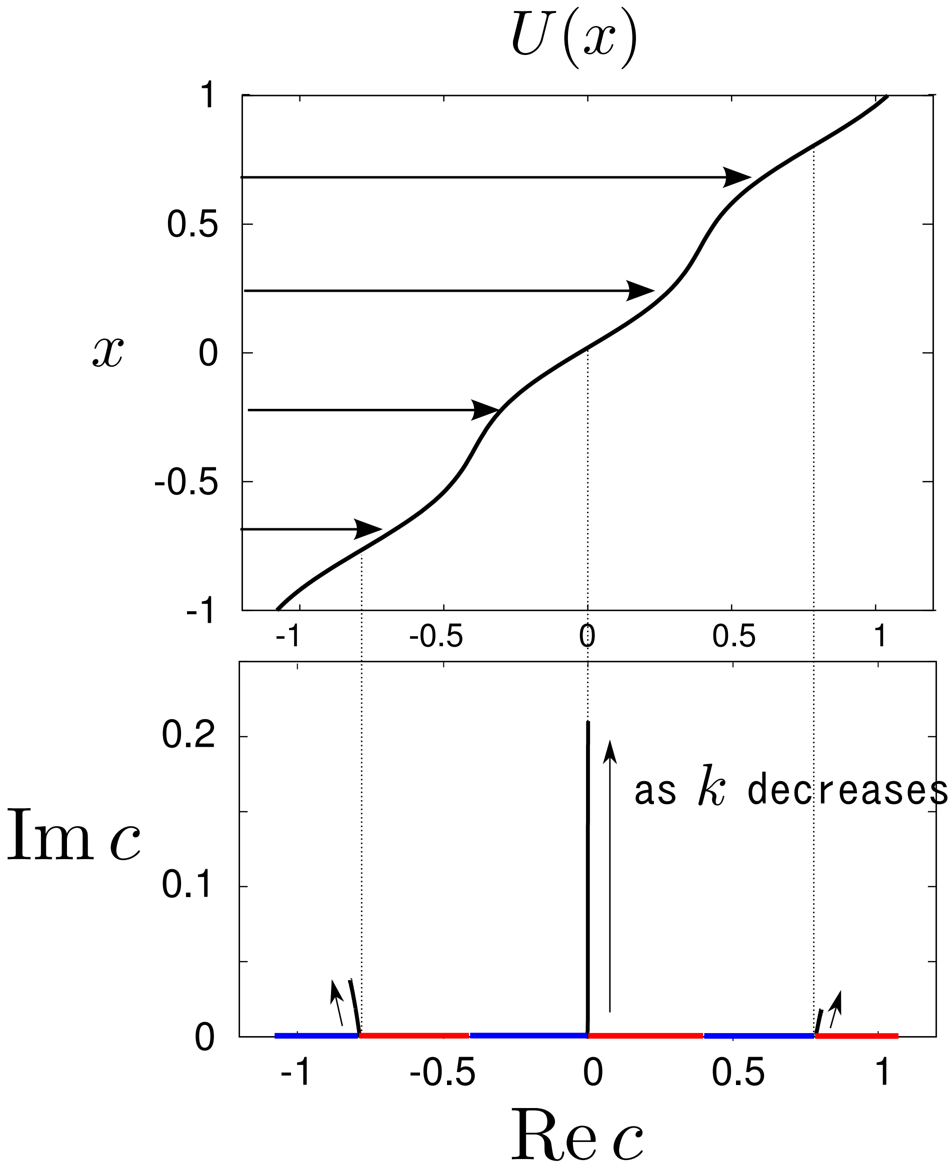
Example 2 (Three inflection points)

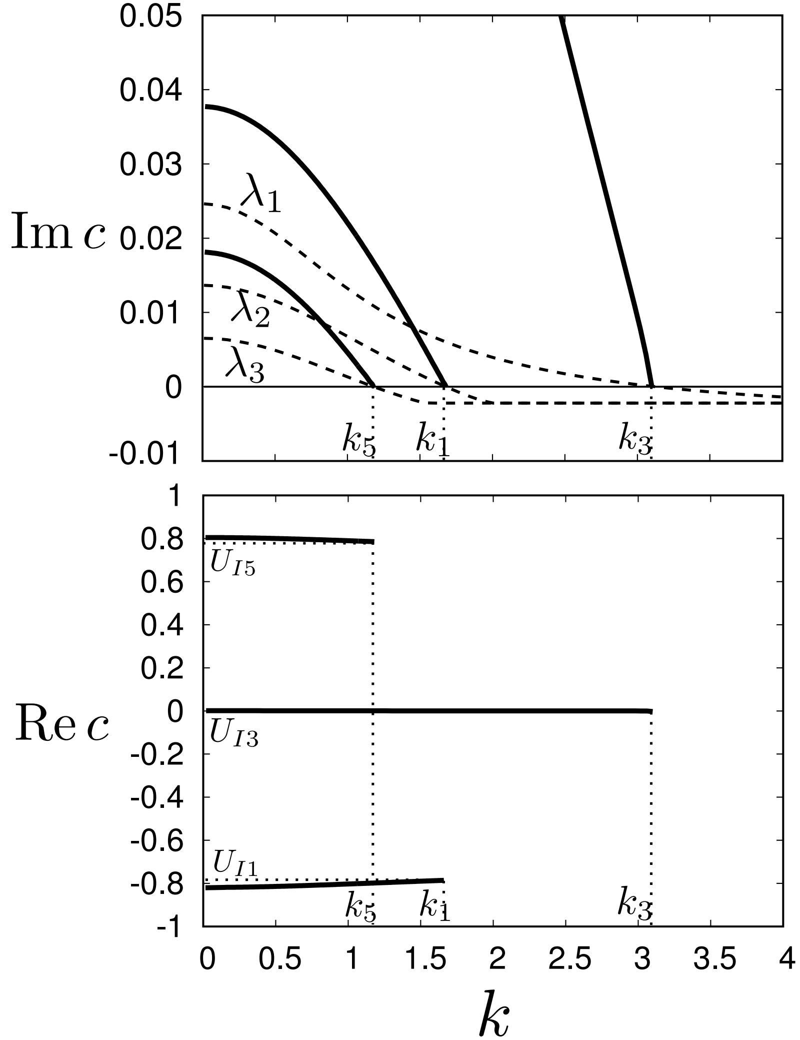
$$U(x) = x + 5x^3 + 1.62\tanh[4(x-0.5)], \quad x \in [-1, 1]$$



Example 3 (Five inflection points)

$$U(x) = x - 0.02 + \sin[8(x-0.02)]/16, \quad x \in [-1, 1]$$





Ideal MHD stability of gravitating plasma slab

Eigenvalue problem (for interchange instability)

$$\left\{\rho\left[(c-U)^2-U_A^2\right]\xi'\right\}'-\left\{k^2\rho\left[(c-U)^2-U_A^2\right]+\rho' g_0\right\}\xi=0,\quad \xi(-L)=\xi(L)=0$$

where $U(x)=\boldsymbol{k}\cdot\boldsymbol{U}(x)/k$ and $U_A(x)=\boldsymbol{k}\cdot\boldsymbol{B}(x)/k\sqrt{\rho\mu_0}$.

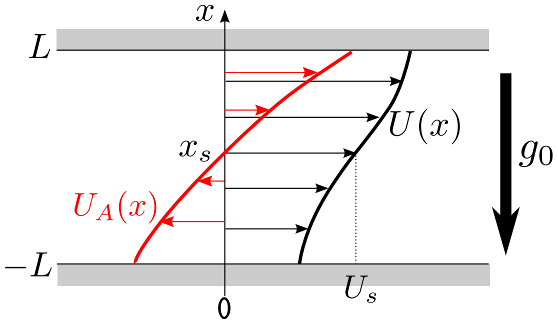
Theorem: Suppose that

1) there is only one resonant surface $x=x_s$ which satisfies $U_A(x_s)=0$
2) $|U_A|>|U-U_s|$ for all x, where $U_s=U(x_s)$.

The equilibrium is spectrally stable if and only if

$$Q=\int_{-L}^{L}\left\{-\rho\left[U_A^2-(U-U_s)^2\right]\left(|\xi'|^2+k^2|\xi|^2\right)+\rho' g_0|\xi|^2\right\}dx,$$

is not positive for all $\xi\in\mathrm{L}^2$.



Shear flow U is always destabilizing.

$$\max_{\xi} \frac{Q}{\|\xi\|^2} > 0 \quad \Leftrightarrow \quad \text{Unstable}$$

Remarks

- $Q = -\delta^2 W = \int \boldsymbol{\xi} \cdot [\mathcal{F} - \rho \boldsymbol{U}_s \cdot \nabla(\boldsymbol{U}_s \cdot \nabla)]\boldsymbol{\xi} d^3x$
- $Q \leq 0$ for Alfvén continuous spectrum

  $Q > 0$ for unstable eigenmodes
- Local stability at resonant surface $x = x_s$ is determined by the modified Suydam criterion (Bondeson et al. 1987)

$$\frac{1}{4} < \left. \frac{\rho' g_0}{\rho(U_A'^2 - U'^2)} \right|_{x=x_s} \quad \Rightarrow \quad \text{Unstable} \quad \left(\begin{array}{l}\text{Infinite sequence of} \\ \text{unstable modes in} \\ \text{the vicinity of } x = x_s\end{array}\right)$$

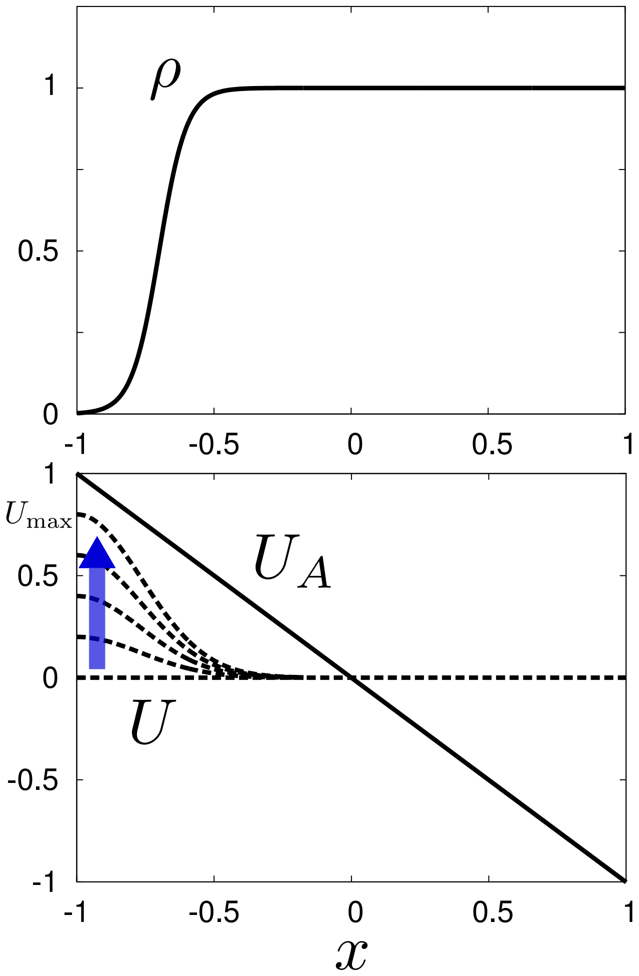
- By adopting the L^2-norm $\|\xi\|^2 = \int |\xi|^2 dx$, the Alfvén continuous spectrum is removed from this variational problem.

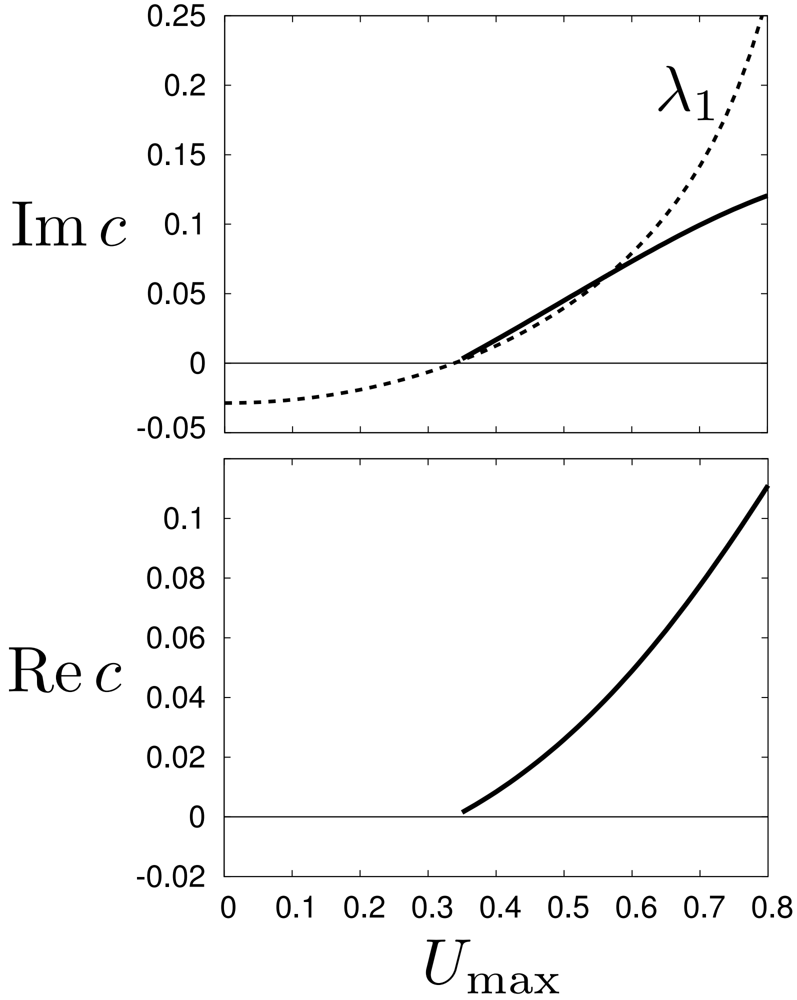
  For the static case (i.e., Newcomb equation), Tokuda & Watanabe (1997) proposed the same idea and employed the weighted-norm $\|\xi\|^2 = \int (x - x_s)^2 |\xi|^2 dx$ in MARG1D and MARG2D codes.

Example (global interchange instability triggered by shear flow)

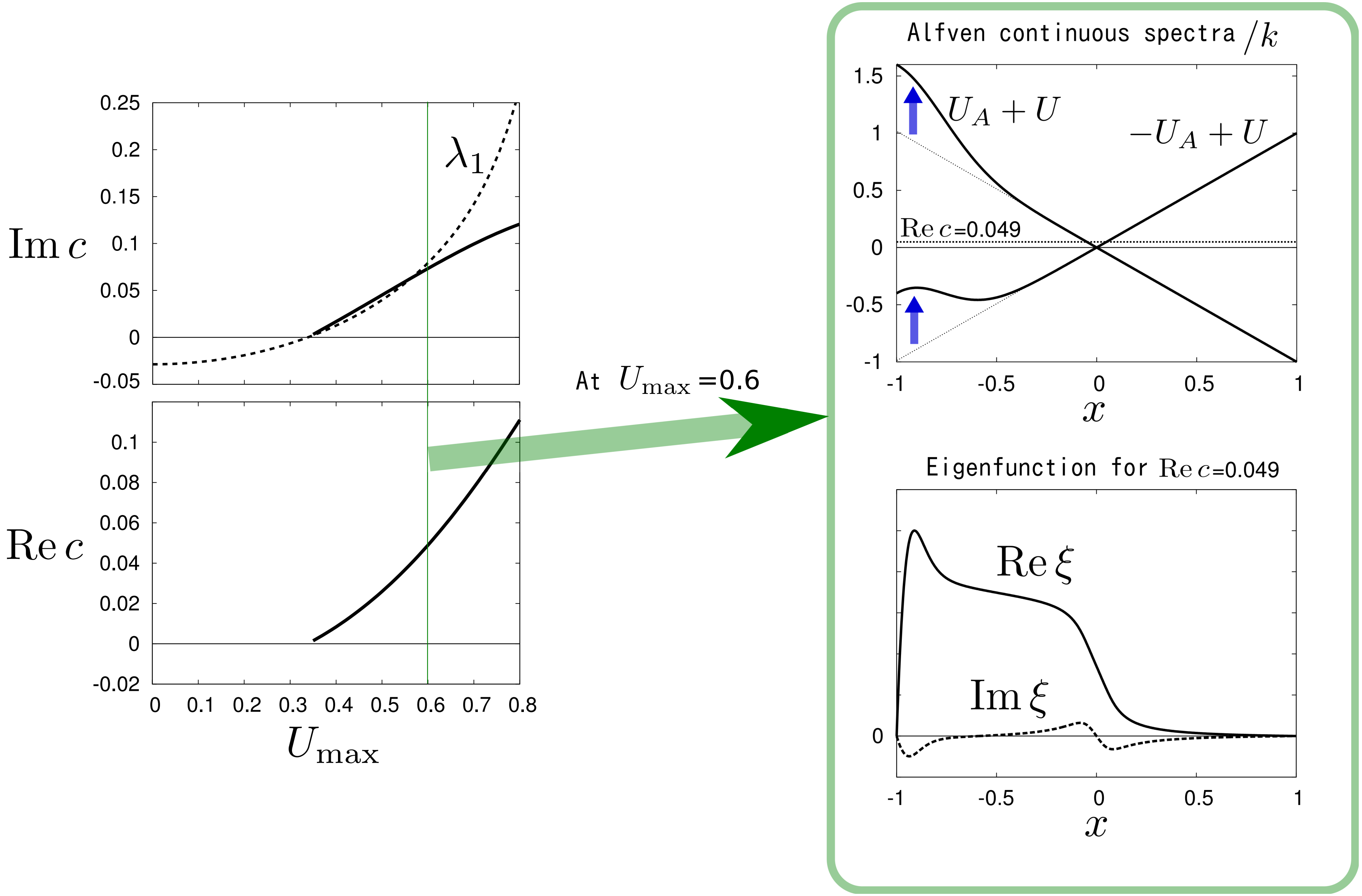
Equilibrium (Suydam stable)

$g_0 = 0.2,\ k = 1.0$





$\lambda_1 = \max Q/\|\xi\|^2 > 0 \Leftrightarrow$ Instability $\mathrm{Im}\, c > 0$



Stable interchange mode (or gravito-Alfvén wave) is Doppler-shifted by shear flow and destabilized by Alfvén resonance.

Summary

- For the stability of shear flow, we have shown that the variational method can be further improved so as to give the necessary and sufficient condition.

$$\boxed{\max \frac{Q}{\|\xi\|^2} > 0 \Leftrightarrow \text{Unstable}}$$

  - Hydrodynamic stability of inviscid parallel shear flow (Rayleigh equation)
  - Ideal MHD stability of gravitating plasma slab

- We can prove instability by finding some test function (virtual displacement) that makes the quadratic form Q positive, which is analytically and numerically feasible without knowing the rigorous solutions of the equation.
- The singularity at the stability boundary can be removed by choosing an appropriate norm $\|\xi\|$ for the functional space.
- We can determine the stability more efficiently and accurately than directly solving the non-selfadjoint and singular eigenvalue equation.
- This variational approach is expected to be applicable to other hydrodynamic stability problems.