Beta limit of MHD equilibrium with pressure anisotropy in magnetospheric configuration


Brief Summary

- Effects of pressure anisotropy on the properties of magnetospheric, axisymmetric toroidal equilibrium with only poloidal magnetic field is investigated.

- Beta limit increases as the pressure anisotropy becomes weak.

- A very high beta, isotropic pressure equilibrium is found where the plasma separates into two torus.
Plasma confinement experiment in magnetospheric configuration (internal coil system) has been conducted

[Z. Yoshida et al., Plasma Fusion Res. 1, 008 (2006).]

Previous studies:
- Numerical study by kinetic calculation of pressure, with magnetic field line connected to the central planet
  [C. Z. Cheng, J. Geophys. Res. 97, 1497 (1992).]
- Analytic solution in point dipole magnetic field
- Numerical studies by using CGL pressure tensor, with internal ring current
  [M. Furukawa, 68th Annual Meeting, Physical Society of Japan, 26aEA–9.]

We study properties of MHD equilibrium with anisotropic pressure in a simplified situation (e.g., a filament approximation of coil current), especially by focusing on diamagnetic current
Extended Grad–Shafranov equation (1)

- Static MHD equilibrium equation using pressure tensor is given by
  \[
  \begin{align*}
  \mathbf{J} \times \mathbf{B} &= \nabla \cdot \mathbf{P} \\
  \mu_0 \mathbf{J} &= \nabla \times \mathbf{B} \\
  \nabla \cdot \mathbf{B} &= 0
  \end{align*}
  \]

- We adopt the CGL pressure tensor in this study:
  \[P = p_{||} \mathbf{b} \mathbf{b} + p_{\perp} (1 - \mathbf{b} \mathbf{b})\]
  \[\mathbf{b} : \text{unit vector along magnetic field}\]

- For axisymmetric toroidal plasma, magnetic field can be expressed as
  \[\mathbf{B} = I \nabla \phi + \nabla \psi \times \nabla \phi\]

- in cylindrical coordinates \((R, \phi, Z)\)

- We obtain, from the force balance in toroidal direction,
  \[I^*(\psi) := \sigma I \quad \sigma := 1 + \frac{\mu_0 (p_{\perp} - p_{||})}{B^2}\]
Extended Grad–Shafranov equation (2)

- If we assume $p_{\parallel}(\psi, B)$, we obtain, from the force balance along the magnetic field,

$$\frac{\partial p_{\parallel}}{\partial B} \bigg|_\psi + \frac{p_{\perp} - p_{\parallel}}{B} = 0$$

[\text{H. Grad, Phys. Fluids 10, 137 (1967).}]

- Now, we obtain, from the force balance in minor-radius direction,

$$\Delta^* \psi = -\mu_0 \frac{R^2}{\sigma} \frac{\partial p_{\parallel}}{\partial \psi} \bigg|_B - \frac{1}{\sigma^2} I^* I^{*'}(\psi) - \frac{1}{\sigma} \nabla \psi \cdot \sigma$$

$$\Delta^* := R^2 \nabla \cdot \left( \frac{1}{R^2} \nabla \right)$$

$$\left( \sigma := 1 + \frac{\mu_0 (p_{\perp} - p_{\parallel})}{B^2} \right)$$

- Boundary condition adopted here is $\psi = 0$ at infinity

- $p_{\perp}(\psi, B)$ is obtained by assuming a functional form of $p_{\parallel}(\psi, B)$

- $I^* \equiv 0$ is used assuming only poloidal magnetic field
Assumptions for pressure (1)

- Assuming

\[
\begin{align*}
\lbrack p_{\parallel}(\psi, B) &= n(\psi, B)T_{\parallel}(\psi) \quad \text{substitute} \quad \frac{\partial p_{\parallel}}{\partial B} \bigg|_{\psi} + \frac{p_{\perp} - p_{\parallel}}{B} = 0 \\
n(\psi, B) &= \bar{n}(\psi) \left( \frac{B}{B_n(\psi)} \right)^{1-\lambda(\psi)} \quad \begin{cases} 
\lambda(\psi) := \frac{T_{\perp}(\psi)}{T_{\parallel}(\psi)} \\
B_n(\psi) : \text{integration constant}
\end{cases}
\end{align*}
\]

- and substitute into the force balance along magnetic field, we obtain

\[
n(\psi, B) = \bar{n}(\psi) \left( \frac{B}{B_n(\psi)} \right)^{1-\lambda(\psi)}
\]

- Therefore,

\[
\begin{align*}
\lbrack p_{\perp}(\psi, B) &= \bar{p}(\psi) \left( \frac{B}{B_n(\psi)} \right)^{1-\lambda(\psi)} \\
p_{\parallel}(\psi, B) &= \frac{1}{\lambda(\psi)} p_{\perp}(\psi, B)
\end{align*}
\]

- where,

\[
\bar{p}(\psi) := \bar{n}(\psi)T_{\perp}(\psi)
\]
We further assume the following:
- $\lambda$ is constant
- $B_n(\psi)$ is taken to be the same
- as $B$ on the outer midplane

Profile is assumed as

$$\bar{p}(\psi) \propto (\psi - \psi_1)(\psi - \psi_2)$$

When we compare equilibria by changing $\lambda$, we keep

$$p = \frac{1}{3}(2p_\perp + p_\parallel)$$

unchanged on the outer midplane
Low beta \( \beta_0 = 10^{-10} \)

\[ \lambda = 0.1 \]
\[ \beta_{\text{local}}^\max = 1.81 \times 10^{-6} \]

\[ \psi^{(\text{blue})}, \frac{p_{\text{para}} + 2p_{\text{perp}}}{3}^{(\text{pink})}, \beta^{(\text{gradation})} \]

- Because of \( \frac{\partial p_{\parallel}}{\partial B} \bigg|_{\psi} + \frac{p_{\perp} - p_{\parallel}}{B} = 0 \), on the same \( \psi \) (magnetic field line),
  - \( \lambda < 1 : p_{\parallel} \), and thus \( p_{\perp} \) also, becomes large at larger \( B \) (inner region of ring current)
  - \( \lambda > 1 : p_{\parallel} \), and thus \( p_{\perp} \) also, becomes large at smaller \( B \) (outer region of ring current)
Equilibrium beta limit

- Numerically obtained beta limit is plotted

\[ \beta_{\text{local}}^{\text{max}} \]: local maximum beta

\[ \langle \beta \rangle \]: volume averaged beta

- \( \beta \) limit seems to be maximum at \( \lambda = 1 \), and decreases more rapidly as \( \lambda \) increases

- Difference between \( \beta_{\text{local}}^{\text{max}} \) and \( \langle \beta \rangle \) is larger at \( \lambda > 1 \)
High beta, two torus equilibrium at $\lambda = 1$

- At $\lambda = 1$, a very high beta equilibrium is obtained, where the plasma separates into two torus

$\langle \beta \rangle = 23.2$