Equilibrium and stability of toroidal plasmas with flow in high-beta reduced MHD

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Equilibrium with flow in extended MHD models of fusion plasmas

- **Equilibrium with flow in fusion plasmas**

  - In improved confinement modes of magnetically confined plasmas, equilibrium toroidal and poloidal flows play important roles like the suppression of instability and turbulent transport.

- **Pressure anisotropy in plasma flow**

  - Plasma flows driven by neutral beam injection indicate strong pressure anisotropy.

- **Small scale effects in MHD equilibria**

  - Equilibrium models with small scale effects may be suitable for modeling steady states of improved confinement modes that have steep plasma profiles and for initial states of multi-scale simulation.

  - Two-fluid equilibria with flow and pressure anisotropy was studied for the case of cold ions [Ito, Ramos and Nakajima, PoP **14**, 062502 (2007)].

  - However, finite ion Larmor radius effects may be relevant for high-temperature plasmas in magnetic confinement fusion devices.

$\rho_i = d_i\sqrt{\beta_i}$, ρ_i : ion Larmor radius, d_i : ion skin depth, $\beta_i \equiv p_i \big/ \left(B_0^2 / \mu_0 \right)$

- Equilibrium with flow in reduced MHD models

  - Fluid moments in collisionless, magnetized plasmas are simplified
  - Grad-Shafranov type equilibrium equations can be easily derived even in the presence of flow and several small scale effects.
  - Basic physics of flow and non-ideal effects can be investigated.
  - Reduced equilibrium models
    - ➢ Two-fluid MHD, FLR, poloidal Alfvenic flow
      [Ito, Ramos and Nakajima, PFR **3**, 034 (2008)]
    - ➢ MHD, poloidal sonic flow
      [Ito, Ramos and Nakajima, PFR **3**, 034 (2008)]
      [Ito and Nakajima, PPCF **51**, 035007 (2009)]
    - ➢ Two-fluid MHD, FLR, poloidal sonic flow, isotropic pressure
      [Ito and Nakajima, AIP Conf. Proc. 1069, 121 (2008)]
      [Raburn and Fukuyama, PoP **17**, 122504 (2010)]
      assumed adiabatic pressure and parallel heat flux was neglected
    - ➢ Two-fluid MHD, FLR, poloidal sonic flow, anisotropic pressure
      [Ito and Nakajima, NF **51**, 123006 (2011)]
      Parallel heat flux with fluid closure is taken into account.

Reduced two-fluid equilibria with poloidal-sonic flow

- Compressible high-β tokamak and slow dynamics orderings

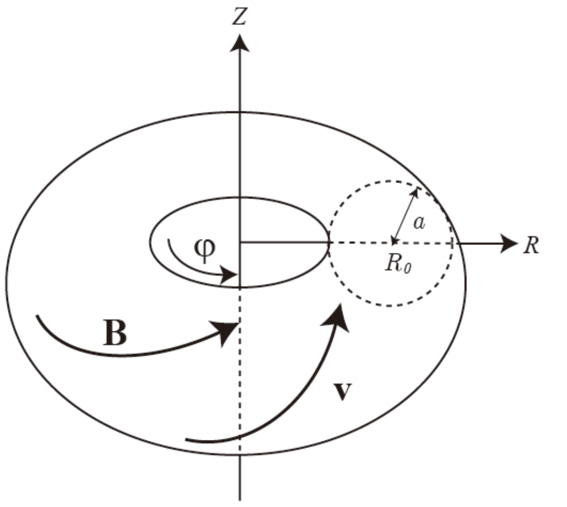
  - Large aspect ratio and high-β tokamak

$$\varepsilon \equiv a/R_0 \ll 1, \quad B_p \sim \varepsilon B_0, \quad p \sim \varepsilon\left(B_0^2/\mu_0\right), \qquad \left|\nabla_{\parallel}\right| \sim 1/R_0, \quad \left|\nabla_{\perp}\right| \sim 1/a$$

a, R_0 : minor and major radii of a torus

B_p, B_0 : poloidal and toroidal magnetic fields

p : pressure



  - Weak compressibility

$$\nabla \cdot \mathbf{v}_{MHD} \sim \varepsilon v_{MHD}/a, \quad \left(\boldsymbol{v} \equiv \boldsymbol{v}_{MHD} + \boldsymbol{v}_{di}\right)$$

The fast magnetosonic wave is eliminated.

  - Flow velocity for slow dynamics

$$v \sim v_{MHD} \sim v_{di} \sim \delta v_{th}, \quad \left|\nabla \cdot \Pi^{gv}\right| \sim \delta^2 \left|\nabla p\right| \sim \delta^2 m_i n v_{th}^2 \quad q \sim pv \sim \delta p v_{th}$$

$\delta \equiv \rho_i / a \ll 1, \quad \rho_i$: ion Larmor radius

- Flow velocity comparable to the poloidal sound velocity

$$\rho v^2 \sim \left(B_p/B_0\right)^2 \gamma p \sim \underline{\varepsilon^3}\left(B_0^2/\mu_0\right) \quad \Rightarrow \quad \delta \sim \varepsilon$$

  - Transition between sub- and super-poloidal-sonic flow appears.
  - Higher-order terms should be taken into account.

- Strong pressure anisotropy: $\left|p_{\parallel} - p_{\perp}\right| \sim p$
- Parallel heat flux can not be neglected: $\left|q_{\parallel} \sim q_{\perp}\right| \sim pv$

- Equilibrium equations in extended-MHD

- Fluid-moment equations for magnetized collisionless plasmas [Ramos, PoP **12** 052102 (2005)]

- Electron inertia is neglected: $m_e \approx 0$

- Two-fluid equilibrium equations with ion FLR

$(\lambda_H, \lambda_i) = (0,0)$	$\Rightarrow$	Single-fluid MHD
$= (1,0)$	$\Rightarrow$	Hall MHD
$= (1,1)$	$\Rightarrow$	FLR two-fluid MHD

$$\nabla \cdot (n\mathbf{v}) = 0, \quad \nabla \times \mathbf{E} = 0, \quad \mu_0 \mathbf{j} = \nabla \times \mathbf{B}, \quad \mathbf{E} \equiv -\nabla\Phi,$$

$$m_i n \mathbf{v} \cdot \nabla \boldsymbol{v} = \mathbf{j} \times \mathbf{B} - \sum_{s=i,e} \left[\nabla p_{s\perp} + \mathbf{B} \cdot \nabla \left(\frac{p_{s\|} - p_{s\perp}}{B^2} \mathbf{B} \right) \right] - \lambda_i \nabla \cdot \Pi_i^{gv},$$

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \frac{\lambda_H}{ne} \left\{ \mathbf{j} \times \mathbf{B} - \left[\nabla p_{e\perp} + \mathbf{B} \cdot \nabla \left(\frac{p_{e\|} - p_{e\perp}}{B^2} \mathbf{B} \right) \right] \right\},$$

$$\mathbf{v} = v_\| \mathbf{b} + \mathbf{v}_\perp, \quad \mathbf{b} \equiv \mathbf{B} / B$$

- FLR effect (λ_i): ion gyroviscosity $\quad \Pi_i^{gv} \sim \varepsilon^2 p$

- Two-fluid effects (λ_H): Hall current and electron pressure

- Strong pressure anisotropy: $p_{\|} - p_{\perp} \sim p$
- Parallel and perpendicular heat flux: $q_{\|} \sim q_{\perp} \sim pv$

- Equations for anisotropic ion and electron pressures

$$\lambda_{i\|} = 0 \quad \Rightarrow \quad \text{adiabatic ion pressure}$$
$$= 1 \quad \Rightarrow \quad \text{ion pressure with parallel heat flux}$$

$$\mathbf{v}\cdot\nabla p_{i\perp} + 2p_{i\perp}\nabla\cdot\mathbf{v} - p_{i\perp}\mathbf{b}\cdot(\mathbf{b}\cdot\nabla\mathbf{v}) + \lambda_{i\|}\nabla\cdot\left(q_{iT\|}\mathbf{b}\right) + \lambda_{i}\nabla\cdot\boldsymbol{q}_{iT\perp} \simeq 0,$$

$$\frac{1}{2}\mathbf{v}\cdot\nabla p_{i\|} + \frac{1}{2}p_{i\|}\nabla\cdot\mathbf{v} + p_{i\|}\mathbf{b}\cdot(\mathbf{b}\cdot\nabla\mathbf{v}) + \lambda_{i\|}\nabla\cdot\left(q_{iB\|}\mathbf{b}\right) + \lambda_{i}\nabla\cdot\boldsymbol{q}_{iB\perp} - 2\lambda_{i}\boldsymbol{q}_{iB\perp}\cdot(\mathbf{b}\cdot\nabla\mathbf{b}) \simeq 0,$$

- Perpendicular (diamagnetic) heat flux: $\boldsymbol{q}_{s\perp} \equiv \boldsymbol{q}_{sB\perp} + \boldsymbol{q}_{sT\perp} \quad (s = i, e)$

$$\mathbf{q}_{sB\perp} \equiv \frac{m_s}{2}\int\left(v_{\|} - \bar{v}_{\|}\right)^2\left(\mathbf{v}_{\perp} - \bar{\mathbf{v}}_{\perp}\right) f d^3\mathbf{v} \simeq \frac{1}{e_s B}\mathbf{b}\times\left[\frac{1}{2}p_{s\perp}\nabla\left(\frac{p_{s\|}}{n}\right) + \frac{p_{s\|}\left(p_{s\|} - p_{s\perp}\right)}{n}(\mathbf{b}\cdot\nabla\mathbf{b})\right],$$

$$\mathbf{q}_{sT\perp} \equiv \frac{m_s}{2}\int\left(v_{\perp} - \bar{v}_{\perp}\right)^2\left(\mathbf{v}_{\perp} - \bar{\mathbf{v}}_{\perp}\right) f d^3\mathbf{v} \simeq \frac{1}{e_s B}\mathbf{b}\times\left[2p_{s\perp}\nabla\left(\frac{p_{s\perp}}{n}\right)\right]$$

- Parallel heat flux: $\boldsymbol{q}_{s\|} \equiv \boldsymbol{q}_{sB\|} + \boldsymbol{q}_{sT\|} \quad (s = i, e)$

$$\mathbf{q}_{sB\|} \equiv \frac{m_s}{2}\int\left(v_{\|} - \bar{v}_{s\|}\right)^2\left(\mathbf{v}_{\|} - \bar{\mathbf{v}}_{s\|}\right) f d^3\mathbf{v}, \quad \mathbf{q}_{sT\|} \equiv \frac{m_i}{2}\int\left(v_{\perp} - \bar{v}_{s\perp}\right)^2\left(\mathbf{v}_{\|} - \bar{\mathbf{v}}_{s\|}\right) f d^3\mathbf{v}$$

- Parallel heat flux equations for ions $\boldsymbol{q}_{i\|} \equiv \boldsymbol{q}_{iB\|} + \boldsymbol{q}_{iT\|}$

[Ramos, PoP **15** 082106 (2008)]

$$\nabla\cdot\left[\left(\mathbf{v}+\frac{\lambda_i}{neB}\nabla p_{i\perp}\times\mathbf{b}\right)q_{iT\|}\right]+q_{iT\|}\nabla\cdot\left(\mathbf{v}+\frac{\lambda_i}{neB}\nabla p_{i\perp}\times\mathbf{b}\right)$$

$$+\frac{p_{i\|}}{m_i}\mathbf{b}\cdot\nabla\left(\frac{p_{i\perp}}{n}\right)-\frac{p_{i\perp}\left(p_{i\|}-p_{i\perp}\right)}{m_i nB}\mathbf{b}\cdot\nabla B\simeq 0,$$

$$\nabla\cdot\left[\left(\mathbf{v}+\frac{\lambda_i}{neB}\nabla p_{i\perp}\times\mathbf{b}\right)q_{iB\|}\right]+\frac{3p_{i\|}}{2m_i}\mathbf{b}\cdot\nabla\left(\frac{p_{i\|}}{n}\right)\simeq 0,$$

- Kinetic effects in the fourth-order moments are neglected

- Parallel heat flux equations for mass-less electrons, $m_e \approx 0$

$$\mathbf{B}\cdot\nabla\left(p_{e\|}/n\right)=0,\quad \mathbf{B}\cdot\nabla\left[\left(p_{e\|}/p_{e\perp}-1\right)B\right]=0$$

- Electron pressure is obtained from the parallel heat flux equations.
- Electron parallel heat flux is obtained from the pressure equations.

- Reduced equilibrium equations

- Axisymmetric equilibria:

$$\partial/\partial\varphi = 0, \quad \mathbf{B} = \nabla\psi \times \nabla\varphi + I\nabla\varphi \qquad \left[\text{cylindrical geometry } (R,\varphi,Z)\right]$$

- Asymptotic expansions: $f = f_0 + f_1 + f_2 + f_3 + \cdots,\ f_1 \sim \varepsilon f_0,\ f_2 \sim \varepsilon^2 f_0,\ f_3 \sim \varepsilon^3 f_0,$

- Lowest order quantities are functions of ψ_1

$$\Phi_1 = \Phi_1(\psi_1), \quad n_0 = n_0(\psi_1), \quad p_{i\|1} = p_{i1}(\psi_1), \quad p_{i\perp 1} = p_{i\perp 1}(\psi_1),$$
$$p_{e\|1} = p_{e\|1}(\psi_1), \quad p_{e\perp 1} = p_{e\perp 1}(\psi_1), \quad I_1 = I_1(\psi_1)$$

- Higher-order quantities are determined by ψ_1 and ψ_2

$$p_{s\{\|,\perp\}2} = p'_{s\{\|,\perp\}1}\psi_2 + \left(\underline{x/R_0}\right)C_{s\{\|,\perp\}}\left(\psi_1\right) + P_{s\{\|,\perp\}2*}\left(\psi_1\right),$$

Shift from magnetic surfaces

$$\Phi_2 = \Phi_1'\psi_2 + \left(x/R_0\right)C_\Phi\left(\psi_1\right) + \Phi_{2*}\left(\psi_1\right), \qquad n_1 = n_0'\psi_2 + \left(x/R_0\right)C_n\left(\psi_1\right) + n_{1*}\left(\psi_1\right),$$

$$v_\| \equiv \left(x/R_0\right)C_{v\|}\left(\psi_1\right) + v_{\|*}\left(\psi_1\right),$$

$$q_{s\|} \equiv \left(x/R_0\right)C_{sq\|}\left(\psi_1\right) + q_{s\|*}\left(\psi_1\right), \qquad q_{sB\|} \equiv \left(x/R_0\right)C_{sqB\|}\left(\psi_1\right) + q_{sB\|*}\left(\psi_1\right),$$

- ➢ $P_{s\{\|,\perp\}*}, v_{\|*}, \Phi_{2*}, n_{1*}, q_{s\|*}, q_{sB\|*}$ are arbitrary functions of ψ_1
- ➢ $C_{s\{\|,\perp\}}, C_{v\|}, C_\Phi, C_n, C_{sq\|}, C_{sqB\|}, C_I$ are obtained from the equations for $p_{s\{\|,\perp\}2}, v_\|, \Phi_2, n_1, q_{s\{\|,\perp\}}, q_{i\mathbf{B}\{\|,\perp\}}, I_2$

- Poloidal force balance

$$p_{i\perp 1} + p_{i\parallel 1} + \frac{B_0}{\mu_0 R_0} I_1 = \text{const.}, \qquad p_{i\perp 2} + p_{e\perp 2} + \frac{B_0}{\mu_0 R_0} I_2 - \left(\frac{x}{R_0}\right) \sum_{s=i,e} (p_{s\parallel 1} - p_{s\perp 1}) \equiv g_*(\psi_1),$$

$$\sum_{s=i,e} \left[p_{s\perp 3} - \left(\frac{x}{R_0}\right)(p_{s\parallel 2} - p_{s\perp 2}) + \frac{1}{2}\left(\frac{x}{R_0}\right)^2 \left(C_{s\parallel} + C_{s\perp} - p_{s\parallel 1} + p_{s\perp 1}\right) \right] + \frac{B_0}{\mu_0 R_0} I_3$$

$$+ \frac{I_1}{\mu_0 R_0}\left(I_2 - I_1' \psi_2\right) - g_*' \psi_2 + + F \frac{\left|\nabla \psi_1\right|^2}{2\mu_0} - \lambda_i \left(\chi_v + \chi_q\right) \equiv E_*(\psi_1),$$

- Radial force balance yields the Grad-Shafranov (GS) type equations

$$\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial Z^2}\right)\psi_1 = -\mu_0 R_0^2 \left[\left(\frac{x}{R_0}\right) \sum_{s=i,e} (p'_{s\parallel 1} + p'_{s\perp 1}) + g'_* \right] - \left(\frac{I_1^2}{2}\right)', \qquad \left(x \equiv R - R_0, \quad x \sim a\right)$$

$$\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial Z^2}\right)\psi_2 + \left[\mu_0 R_0^2 \left(\frac{x}{R_0}\right) \sum_{s=i,e} \left(p''_{s\perp 1} + p''_{s\parallel 1}\right) + \mu_0 R_0^2 g''_* + \left(\frac{I_1^2}{2}\right)'' \right]\psi_2 = \frac{1}{R}\frac{\partial \psi_1}{\partial R} + F\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial Z^2}\right)\psi_1 + F' \frac{\left|\nabla \psi_1\right|^2}{2}$$

$$- \mu_0 R_0^2 \left[E'_* + \left(\frac{x}{R_0}\right) \sum_{s=i,e} \left(P'_{s\perp 2*} + P'_{s\parallel 2*}\right) + \frac{1}{2}\left(\frac{x}{R_0}\right)^2 \sum_{s=i,e} \left(p'_{s\perp 1} + p'_{s\parallel 1} + C'_{s\perp 1} + C'_{s\parallel 1}\right) \right].$$

- GS equation for ψ_2 includes the effect of flow, FLR and pressure anisotropy:

$$F(\psi_1) \equiv \left[V_E - \underline{\left(\lambda_H - \lambda_i\right)} V_{di}\right]\left(V_E - \lambda_H V_{di}\right) + \underline{\sum_{s=i,e} \frac{p_{s\parallel 1} - p_{s\perp 1}}{B_0^2 / \mu_0}}$$

Gyroviscous cancellation (FLR effect)

Pressure anisotropy

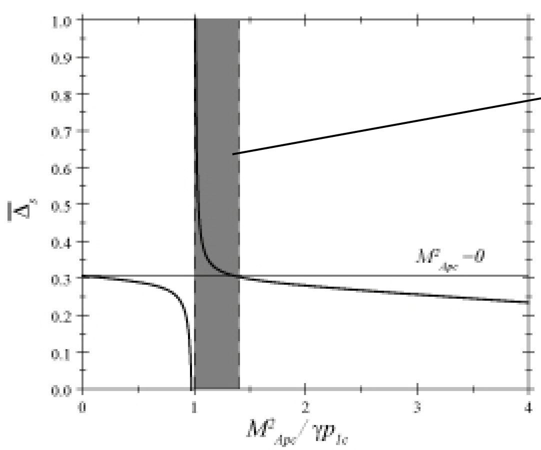
$V_E(\psi_1), V_{di}(\psi_1)$: Poloidal Alfven Mach numbers of the $E \times B$ drift and the ion diamagnetic drift velocities

Analytic solution for single-fluid, isotropic and adiabatic pressure case

[A. Ito and N. Nakajima, Plasma Phys. Control. Fusion **51**, 035007 (2009)]

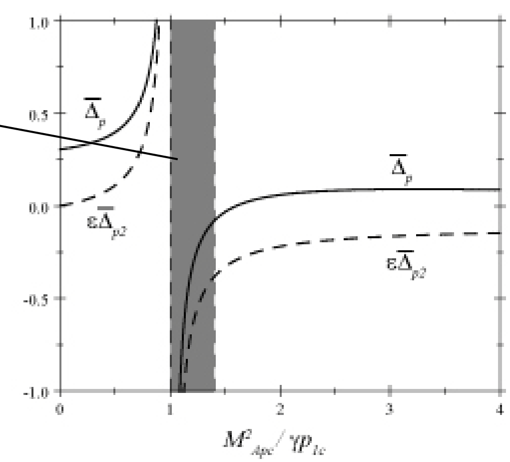
- Magnetic structure is modified to yield a forbidden region and the pressure surface departs from magnetic surface due to poloidal-sonic flow.

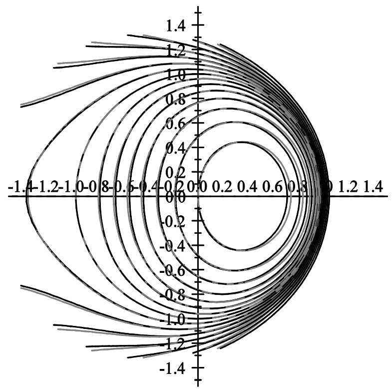
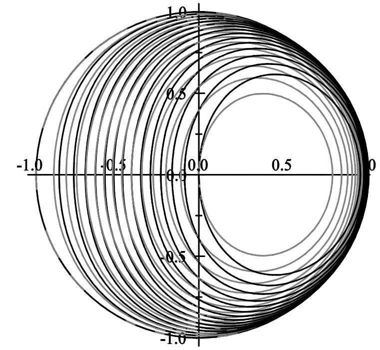
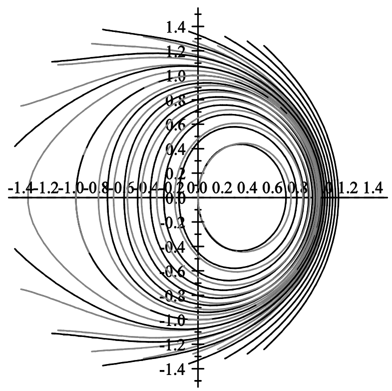
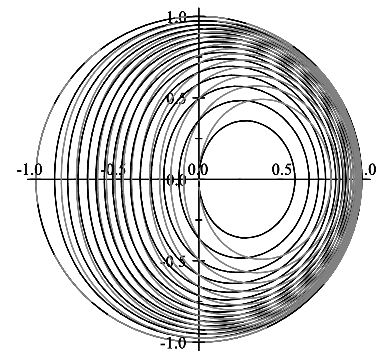
➢ Shift of the magnetic axis from the geometric axis



Beyond beta limit

➢ Shift of the pressure maximum



	Magnetic surfaces (gray: static equilibrium)	Pressure surfaces (gray: magnetic surfaces)
Sub-poloidal-sonic flow $\left(\gamma p_{1c}/M_{Apc}^2 = 0.5\right)$		
Super-poloidal-sonic flow $\left(\gamma p_{1c}/M_{Apc}^2 = 2.5\right)$		

- The pressure maximum is shifted outwards for sub-poloidal-sonic flow and inwards for super-poloidal-sonic flow

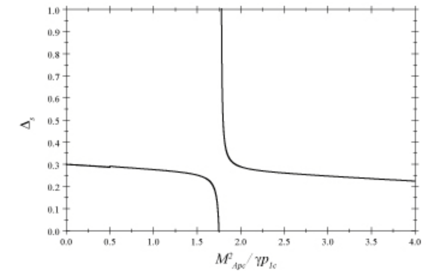
Analytic solution for single-fluid equilibrium with flow and pressure anisotropy: $(\lambda_H, \lambda_i) = (0, 0)$

[A. Ito and N. Nakajima, J. Phys. Soc. Jpn **82** (2013) 064502.]

- Anisotropic, double adiabatic ion pressure, $\lambda_{i||} = 0$
  - ➢ Singularity

    $M_{Apc}^2 = 3p_{i||1c} + p_{e||1c}$ (slow magnetosonic wave)
  - ➢ Qualitatively same as the isotropic case

Shift of the magnetic axis due to flow

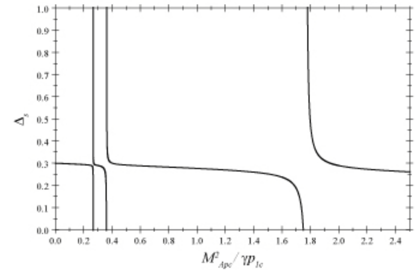


- Anisotropic ion pressure in the presence of the parallel heat flux, $\lambda_{i||} = 1$
  - ➢ Singularity

$$M_{Apc}^2 = \frac{1}{2}\left(6p_{i||1c} + p_{e||1c} \pm \sqrt{24p_{i||1c}^2 + p_{e||1c}^2}\right)$$

(slow magnetosonic and ion acoustic waves)

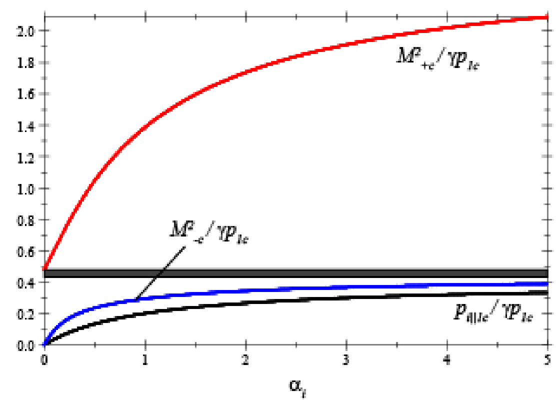
$M_{Apc}^2 = p_{i||1c}$ (ion acoustic wave)



- Complicated characteristics in the region around the poloidal sound velocity due to pressure anisotropy and the parallel heat flux have been found
- Dependences of poloidal sonic singularity on the pressure anisotropy for ions α_i

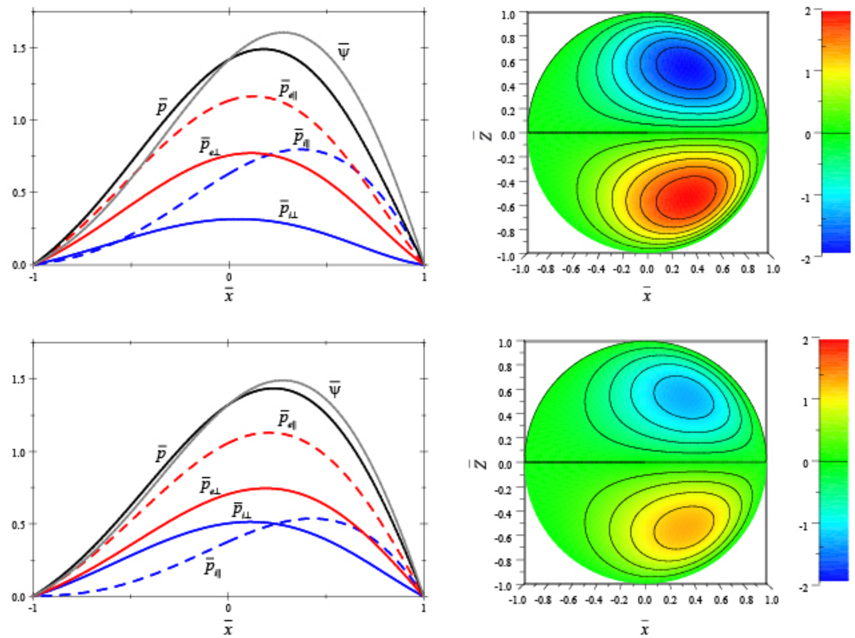
$$p_{i\|1c} = \alpha_i p_{i\perp 1c},$$

$$p_{1c} = \frac{1}{2}\sum_{s=i,e}\left(p_{s\|1c} + p_{s\perp 1c}\right) \text{ is fixed.}$$



In the shaded region ,the equilibrium is regular and the poloidal Mach number lies between the two singular points for all values of α_i

Profiles of pressures and the magnetic flux in the midplane (Left) and the radial component of the diamagnetic current in a poloidal cross section (right) for the poloidal Mach number 0.45, α_i=2.0 (top) and 0.75 (bottom).



- Perpendicular (diamagnetic) current

$$\mathbf{j}_\perp \cdot \nabla \psi \simeq \frac{1}{B_0} \sum_{s=i,e} \left[C_{s\perp} - (\underline{p_{s\|1} - p_{s\perp 1}}) \right] \{x, \psi_1\}$$

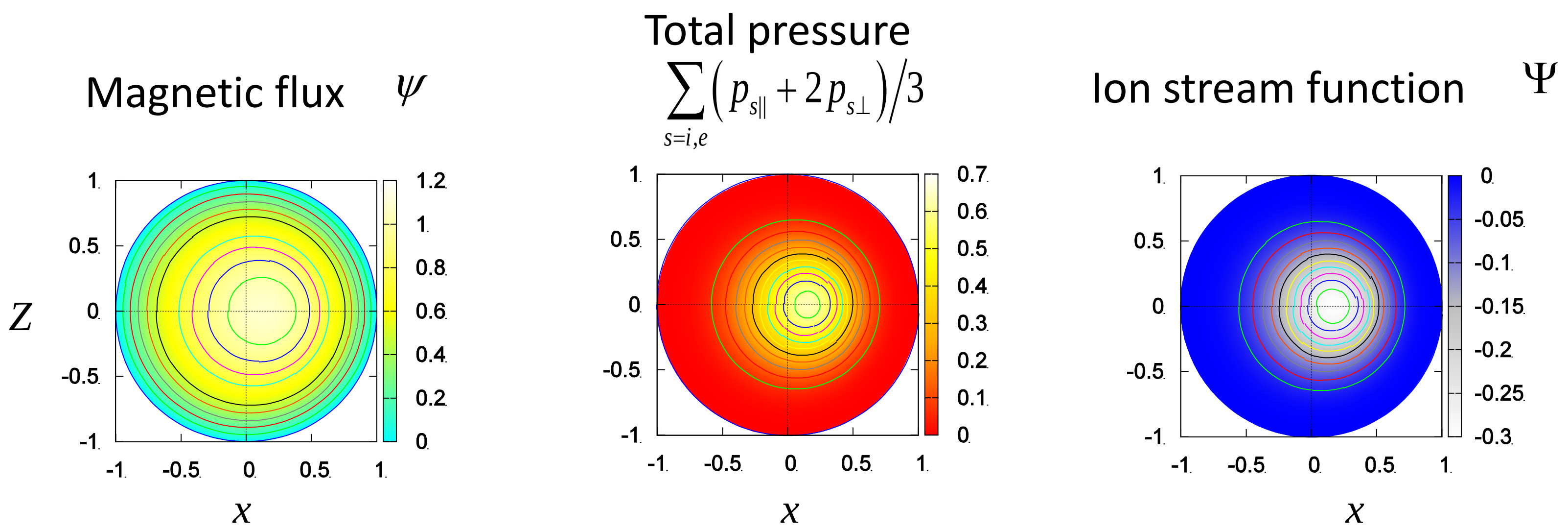
Decreasing a_i with the poloidal Mach number fixed in the shaded region moves the equilibrium from the super poloidal sonic region to the sub poloidal sonic one similarly to how increasing the poloidal Mach number at a fixed a_i does.

Numerical solution for the FLR two-fluid model

- Boundary conditions:
  - Circular cross-section,
  - Up-down symmetry

$$\psi_1(1,\theta)=0, \quad \psi_2(1,\theta)=0.$$

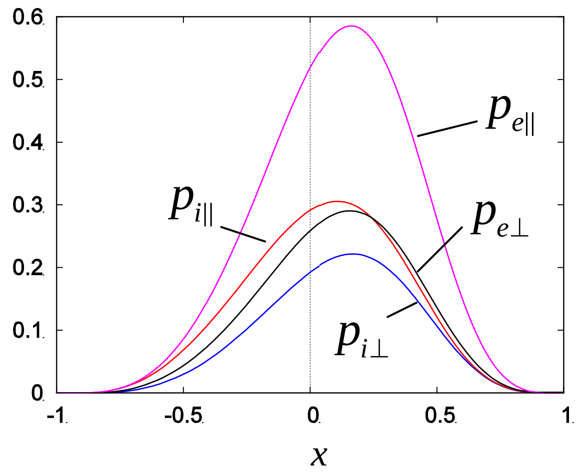
- Finite element method
  - GS Eq. for ψ_1: nonlinear, solved iteratively
  - GS Eq. for ψ_2: linear, solved by substituting ψ_1
- Numerical results for two-fluid equilibria $\left(\varepsilon=0.1, V_{Ec}=\sqrt{0.4\gamma p_{1c}}, V_{dic}=\sqrt{0.2\gamma p_{1c}}\right)$



- Isosurfaces of each quantity do not coincide because of the flow, pressure anisotropy and the two-fluid effects.

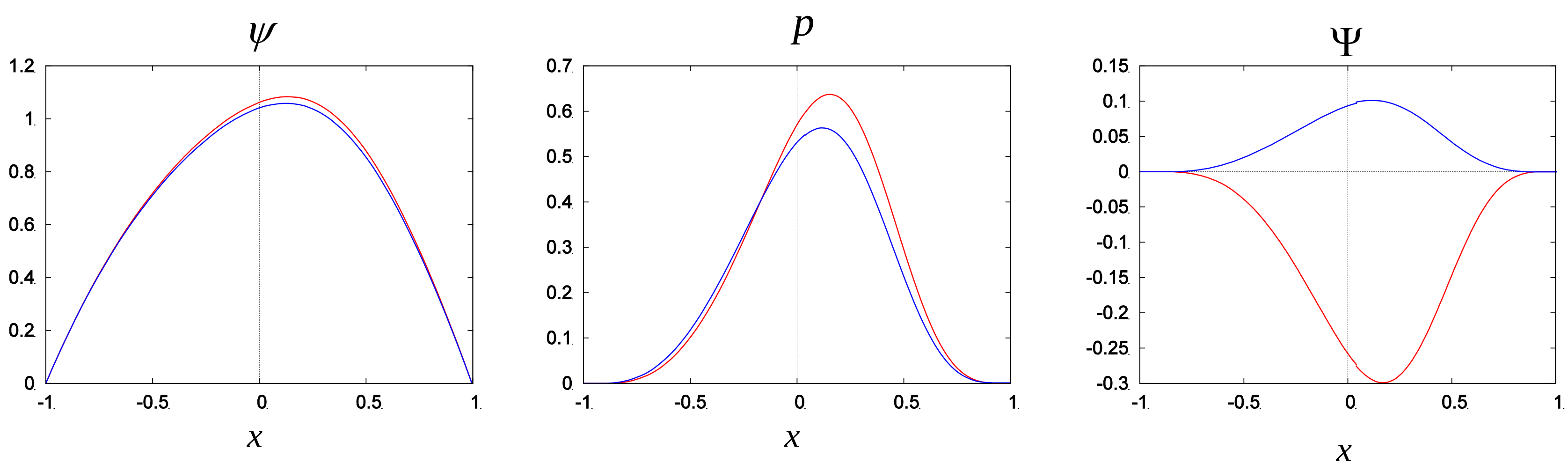
- Pressure profiles in the midplane

$\left(p_{i\perp 1c} = 0.75 p_{e\perp 1c},\ p_{i\|1c} = 1.5 p_{i\perp 1c},\ p_{e\|1c} = 2.0 p_{e\perp 1c} \right)$



Anisotropic pressures for ions and electrons are self-consistently obtained.

- Solutions depend on the sign of $E\times B$ flow compared to that of ion diamagnetic flow



Red: same direction

Blue: opposite direction

Reduced MHD equations for stability of poloidal-sonic flow

- Reduced-MHD equations with $\partial/\partial t \neq 0, \partial/\partial\varphi \neq 0$

  - must include equilibria with poloidal-sonic flow when $\partial/\partial t = 0, \partial/\partial\varphi = 0$

  - require the energy conservation up to $O\left[\varepsilon^3\left(B_0^2/\mu_0\right)\right]$

  - are needed for stability of toroidal equilibria with strong poloidal flow

- Derivation of the reduced equations for single-fluid MHD

We modify the reduced equations found by Strauss [Strauss, 1983] to apply for high-beta plasmas with poloidal-sonic dynamics and non-constant density.

$$\mathbf{v} \equiv \left(1 + x/R_0\right)\nabla U \times \left(\mathbf{B}/B\right) + v_{\parallel}\left(\mathbf{B}/B\right), \quad \mathbf{B} \equiv \mathbf{H} \times \nabla\varphi + I\nabla\varphi,$$

$$\mathbf{H} \equiv \nabla\psi - \frac{1}{\xi}\frac{\partial^2 F}{\partial\varphi\partial\Theta}\nabla\xi\left(R,Z\right) + \xi\frac{\partial^2 F}{\partial\varphi\partial\xi}\nabla\Theta\left(R,Z\right).$$

$$\left[\frac{\partial}{\partial t} + R_0\left(\nabla U \times \nabla\varphi\right)\cdot\nabla\right]\left[\nabla_\perp\cdot\left(\rho R_0\nabla_\perp U\right)\right] + \left\{\rho, \frac{R_0^2\left|\nabla_\perp U\right|^2}{2}\right\} - \left\{R^2 - R_0^2, p\right\}$$

$$+\left[\mathbf{H}\times\nabla\varphi + B_0 R_0\left(1 - \frac{p}{B_0^2/\mu_0}\right)\nabla\varphi\right]\cdot\nabla\left(Rj_\varphi\right) + \mu_0\frac{j_\varphi}{B_0}\frac{\partial p}{\partial\varphi} + \frac{1}{B_0 R_0}\frac{\partial\mathbf{H}}{\partial\varphi}\cdot\nabla p = 0,$$

$$\frac{\partial\psi}{\partial t} = \frac{R^2}{R_0}\nabla\varphi\cdot\left(\nabla U \times \mathbf{H}\right) + B_0\frac{\partial U}{\partial\varphi} - \frac{\partial\widehat{\Phi}}{\partial\varphi},$$

$$\rho\left[\frac{\partial}{\partial t} + R_0\left(\nabla U \times \nabla\varphi\right)\cdot\nabla\right]v_{\parallel} + \frac{R}{B_0 R_0}\left[\left(1 + \frac{p}{B_0^2/\mu_0}\right)\mathbf{H}\times\nabla\varphi + B_0 R_0\nabla\varphi\right]\cdot\nabla p = 0,$$

$$
\begin{aligned}
&\frac{\partial p}{\partial t}+\frac{v_{\parallel}R}{B_0R_0}\left[\left(1+\frac{p}{B_0^2/\mu_0}\right)\mathbf{H}\times\nabla\varphi+B_0R_0\nabla\varphi\right]\cdot\nabla p\\
&+\left(R/BR_0\right)\left[I\nabla U\times\nabla\varphi+\left(\nabla U\cdot\nabla\varphi\right)\mathbf{H}-\left(\mathbf{H}\cdot\nabla U\right)\nabla\varphi\right]\cdot\nabla p\\
&+\gamma p\left[\mathbf{H}\times\nabla\varphi+B_0R_0\left(1-\frac{p}{B_0^2/\mu_0}\right)\nabla\varphi\right]\cdot\nabla\left[\frac{v_{\parallel}R}{B_0R_0}\left(1+\frac{p}{B_0^2/\mu_0}\right)\right]\\
&+\gamma p\nabla\left(R/BR_0\right)\cdot\left[I\nabla U\times\nabla\varphi+\left(\nabla U\cdot\nabla\varphi\right)\mathbf{H}-\left(\mathbf{H}\cdot\nabla U\right)\nabla\varphi\right]\\
&+\frac{\gamma pR}{BR_0}\left[\left(\nabla U\cdot\nabla\varphi\right)\nabla\cdot\mathbf{H}+\mathbf{H}\cdot\nabla\left(\nabla U\cdot\nabla\varphi\right)-\nabla\varphi\cdot\nabla\left(\mathbf{H}\cdot\nabla U\right)+\nabla I\cdot\left(\nabla U\times\nabla\varphi\right)\right]=0,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{R^2}\frac{\partial I}{\partial t}+\left(\nabla U\times\nabla\varphi\right)\cdot\nabla\left(\frac{I^2}{R_0RB}\right)+\frac{I}{R_0RB}\left(\nabla U\cdot\nabla\varphi\right)\nabla\cdot\mathbf{H}+\mathbf{H}\cdot\nabla\left[\frac{I}{R_0RB}\left(\nabla U\cdot\nabla\varphi\right)\right]\\
&+\left(\mathbf{H}\times\nabla\varphi\right)\cdot\nabla\left(\frac{\mathbf{H}\cdot\nabla U}{R_0RB}\right)+\frac{\mu_0}{R_0RB}\left(\mathbf{H}\cdot\nabla U\right)\left(\nabla\varphi\cdot\nabla p\right)=0,
\end{aligned}
$$

$$
\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial F}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial^2F}{\partial\Theta^2}=\frac{\mu_0p}{\xi JB_0},\quad J\equiv R_0\left(\nabla\xi\times\nabla\Theta\right)\cdot\nabla\varphi
$$

$$
\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\widehat{\Phi}}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial^2\widehat{\Phi}}{\partial\Theta^2}=\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\mu_0p}{B_0}\frac{\partial U}{\partial\xi}\right)+\frac{1}{\xi}\frac{\partial}{\partial\Theta}\left(\frac{\mu_0p}{B_0}\frac{\partial U}{\partial\Theta}\right),
$$

These equations contain shear Alfven and slow magnetosonic waves.

- Energy conservation

$$
\frac{1}{2}\frac{\partial}{\partial t}\int d^3\mathbf{x}\left\{\left[\rho\left(\left|\nabla U\right|^2+v_{\parallel}^2\right)+\frac{1}{\mu_0R^2}\left(\left|\mathbf{H}\right|^2+I^2\right)+\frac{2p}{\gamma-1}\right]\right\}=0,
$$

can be shown with asymptotic expansions in terms of ε up to $O\left[\varepsilon^3\left(B_0^2/\mu_0\right)\right]$.

Summary

- Reduced equations for two-fluid equilibria with flow
  - Two-fluid equilibria with toroidal and poloidal flow, ion FLR, pressure anisotropy and parallel heat flux have been derived.
- Analytic solution for single-fluid equilibria
  - The solution indicates the modification of the magnetic flux and the departure of the pressure surfaces from the magnetic surfaces due to flow.
  - Complicated characteristics in the region around the poloidal sound velocity due to pressure anisotropy and the parallel heat flux have been found.
- Numerical solution for two-fluid equilibria with ion FLR
  - The isosurfaces of the magnetic flux, the pressure and the ion stream function do not coincide with each other.
  - Pressure anisotropy associated with parallel heat flux has been included in the numerical code.
  - Solutions depend on the direction of $E \times B$ flow compared to that of ion diamagnetic flow.
- Reduced MHD equations for stability of poloidal-sonic Flow
  - We have derived reduced single-fluid MHD equations with time evolution in order to study their stability of equilibria with poloidal-sonic flow
  - We have shown that the energy is conserved up to the order required by the equilibria.