第20回NEXT（数値トカマク）研究会
2015年1月13日（火） – 1月14日（水）　京都テルサ

圧縮性MHDに対する
ロバストな数値計算法の開発

Development of a robust scheme for compressible MHD

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MHD code projects

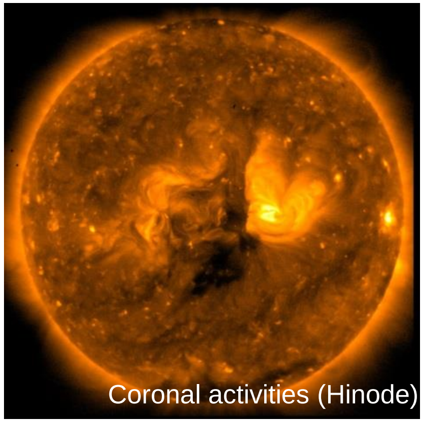
For laboratory plasmas

Project	Developer	MHD Scheme	div B	Grid
(US)				
NIMROD	Sovinec	FEM(2D)+SP(1D) / implicit	Diffusion	Triangular
M3D-C^1	Ferraro, Jardin	C^1 FEM / implicit	Vec. pot.	Triangular
(EU)				
XTOR-2F	Lütjens	FD(1D)+SP(2D) / NK implicit	—	Mag. Flux
(Japan)				
MIPS	Todo	4th FD / 4th RKG	—	Cylindrical
MINOS	Miura	8th Compact FD / 4th RKG	—	Curvilinear

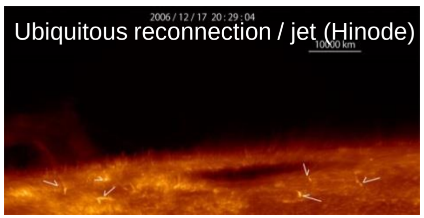
Any information or corrections are appreciated…

- Extended MHD model
- Not designed for shock capturing

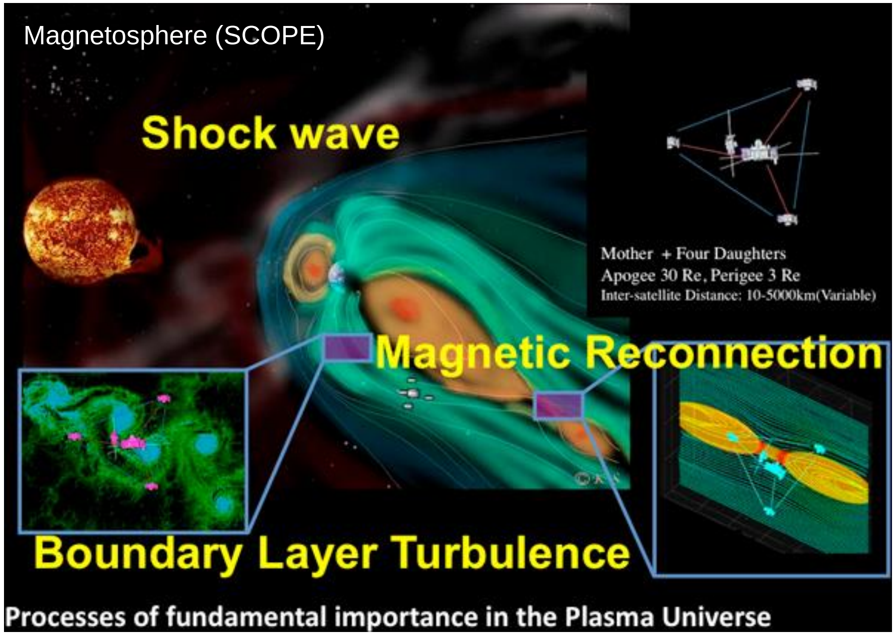
Shocks in space plasmas



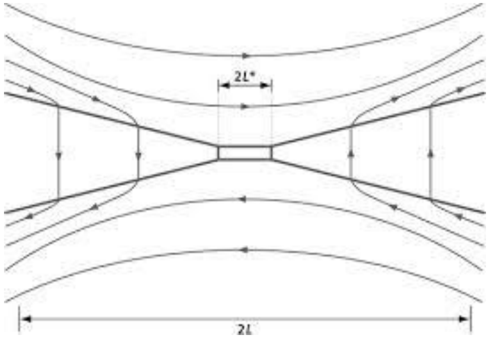
Coronal activities (Hinode)



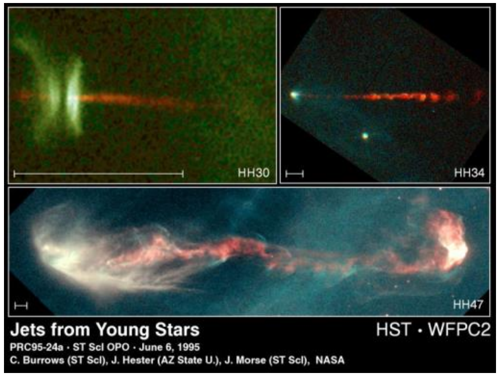
Ubiquitous reconnection / jet (Hinode)

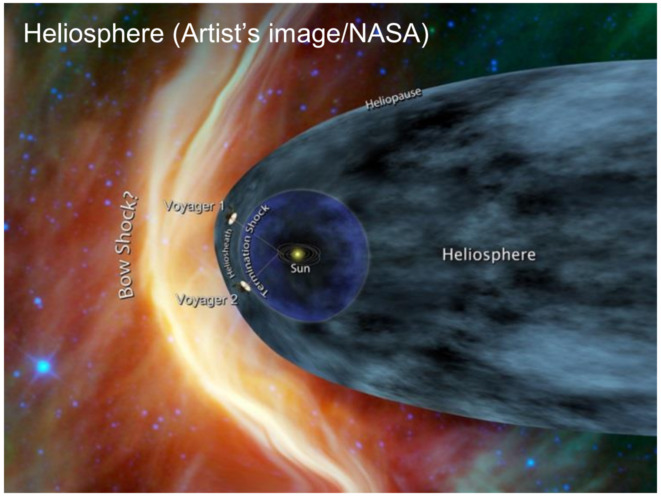


Magnetosphere (SCOPE)

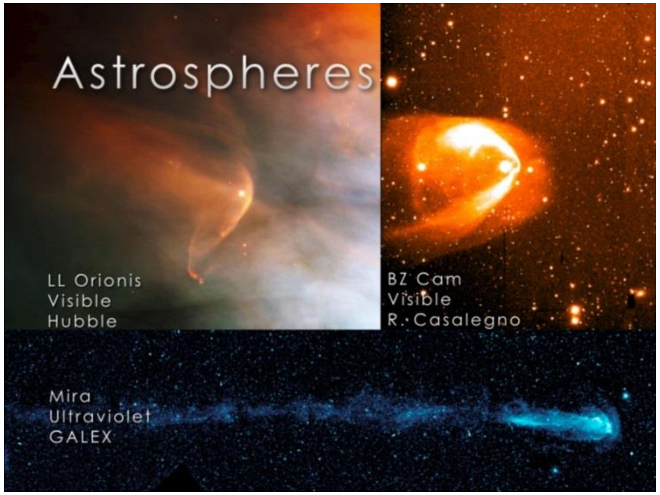


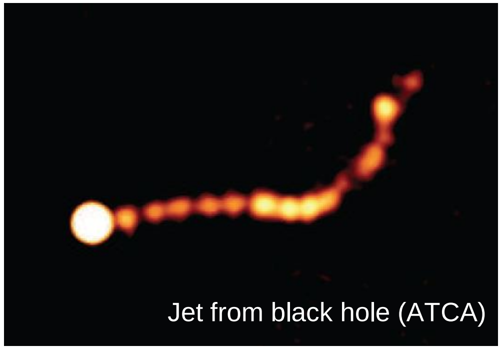
Petschek-type reconnection





Heliosphere (Artist's image/NASA)





Jet from black hole (ATCA)

Motivation and objectives

- In compressible MHD codes for laboratory plasmas, time integration methods have been polished so as to solve stiff problems.
- But, those codes are not designed for shock capturing that may be needed in the near future. (e.g., HiFi code at PSI-Center)
- Shocks and turbulence are universally observed in space. Thus, the development of robust shock capturing schemes has been highly progressed.
- Current status and challenges of the shock capturing scheme for MHD are presented with emphasis on our results.

Compressible MHD equations

- Ideal MHD equations (Non-conservative form)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \quad : \text{continuity equation}$$

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}\right) = -\nabla p + \boldsymbol{J} \times \boldsymbol{B} \quad : \text{equation of motion}$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^{\gamma}}\right) + \boldsymbol{v} \cdot \nabla\left(\frac{p}{\rho^{\gamma}}\right) = 0 \quad : \text{adiabatic equation}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = 0, \ \nabla \cdot \boldsymbol{B} = 0 \quad : \text{induction equation}$$

  - Various non-conservative forms can be obtained using vector identities

Compressible MHD equations

- Ideal MHD equations (Conservative form)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \qquad : \text{mass conservation}$$

$$\frac{\partial (\rho \boldsymbol{v})}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}\boldsymbol{v} + p_T \boldsymbol{I} - \boldsymbol{B}\boldsymbol{B}) = 0 \qquad : \text{momentum conservation}$$

$$\frac{\partial e}{\partial t} + \nabla \cdot [(e + p_T)\boldsymbol{v} - \boldsymbol{B}(\boldsymbol{v} \cdot \boldsymbol{B})] = 0 \qquad : \text{energy conservation}$$

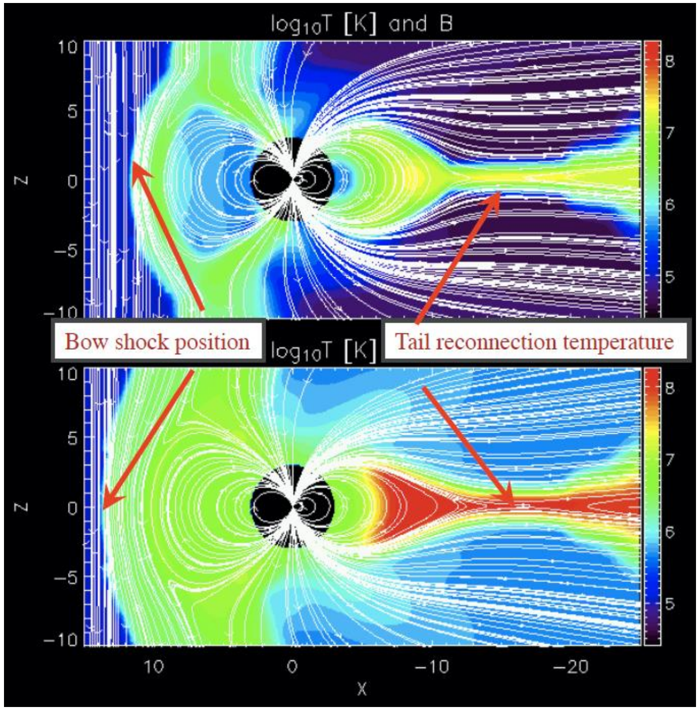
$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \cdot (\boldsymbol{v}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{v}) = 0 \qquad : \text{flux conservation}$$

$$\nabla \cdot \boldsymbol{B} = 0 \,, \quad p = (\gamma - 1)\left(e - \frac{\rho v^2}{2} - \frac{B^2}{2} \right), \quad p_T = p + \frac{B^2}{2}$$

Shock capturing scheme

- Non-conservative scheme
  - Based on non-conservative form
  - Converge to unphysical shock
    Hou-LeFloch [1994]
- Conservative scheme
  - Based on conservative form
  - Converge to physical shock
    Lax-Wendroff [1960]
    Harten [1980]
  - Difficult to preserve positivity

Conservative vs Non-conservative



"Computational Tutorial: MHD", Toth

http://www.lorentzcenter.nl/lc/web/2011/441/presentations/Advanced_Toth.pdf

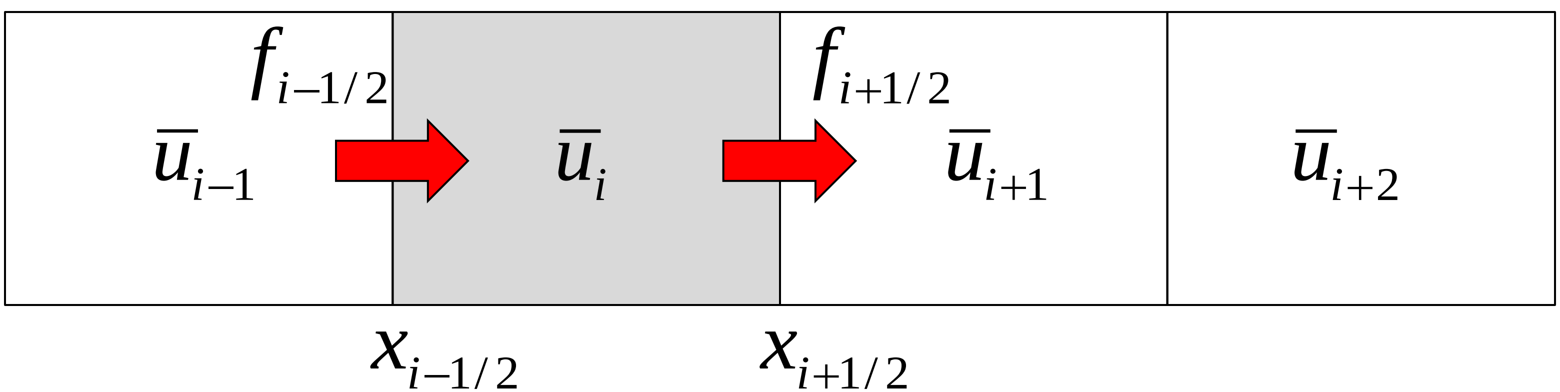
Shock capturing scheme

- Non-conservative scheme
  - Finite difference method
  - Finite element method
- Conservative scheme
  - Finite difference method
    - FD-WENO, Compact FD+LAD, etc.
  - Finite element method
    - RKDG, etc.
  - Finite volume method
    - MUSCL, FV-WENO, etc.

Shock capturing scheme

- 1D finite volume method

$$\frac{\partial u}{\partial t}+\frac{\partial f}{\partial x}=0 \Rightarrow \frac{d}{dt}\bar{u}_i+\frac{f\left(u\left(x_{i+1/2},t\right)\right)-f\left(u\left(x_{i-1/2},t\right)\right)}{\Delta x}=0$$



- Numerical flux

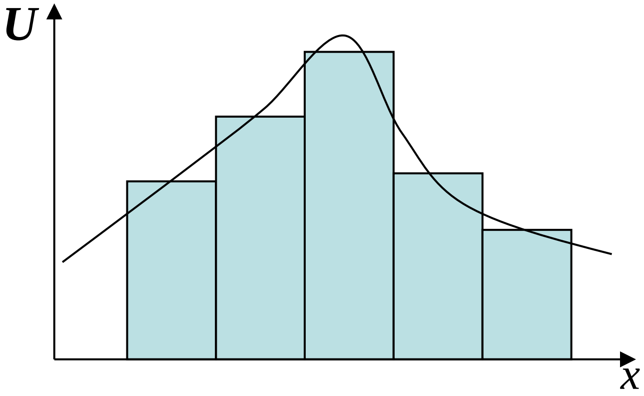
$$f\left(u\left(x_{i+1/2},t\right)\right)\equiv f_{i+1/2}=f\left(\cdots,\bar{u}_{i-1},\bar{u}_i,\bar{u}_{i+1},\bar{u}_{i+2},\cdots\right)$$

Approximate Riemann solver

- Approximate Riemann solver

$$\iint \left(\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} \right) dxdt = \oint \left(\boldsymbol{U}dx - \boldsymbol{F}dt \right) = 0$$

  - Define piecewise constants

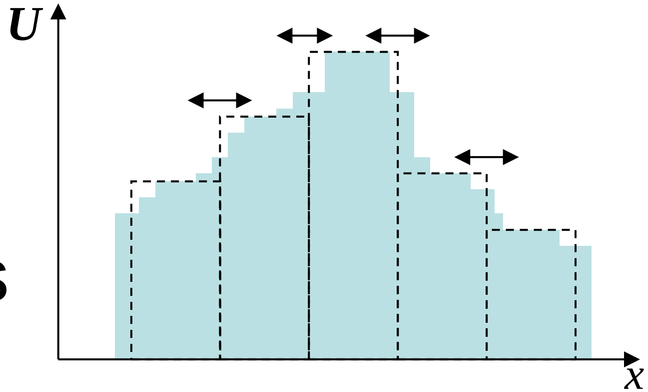


Approximate Riemann solver

- Approximate Riemann solver

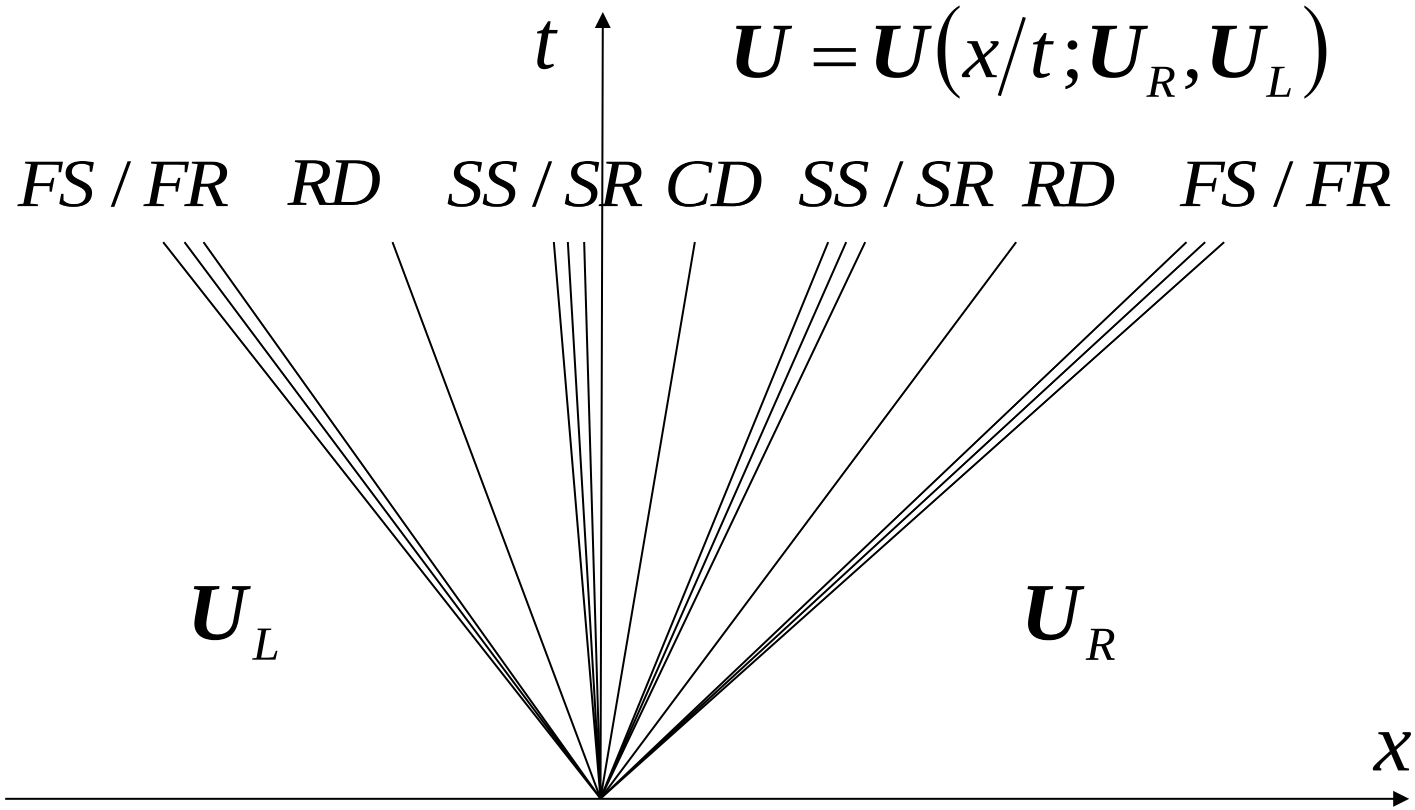
$$\iint\left(\frac{\partial \boldsymbol{U}}{\partial t}+\frac{\partial \boldsymbol{F}}{\partial x}\right)dxdt=\oint\left(\boldsymbol{U}dx-\boldsymbol{F}dt\right)=0$$

  - Define piecewise constants
  - Solve local Riemann problems



Riemann problem

- Riemann problem = Shock tube problem
  - 7-waves can be excited in 1D MHD system (shock, expansion wave, compound wave)

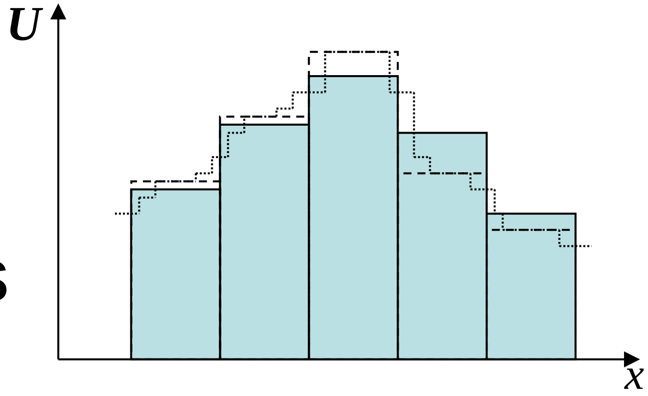


Approximate Riemann solver

- Approximate Riemann solver

$$\iint\left(\frac{\partial \boldsymbol{U}}{\partial t}+\frac{\partial \boldsymbol{F}}{\partial x}\right)dxdt=\oint\left(\boldsymbol{U}dx-\boldsymbol{F}dt\right)=0$$

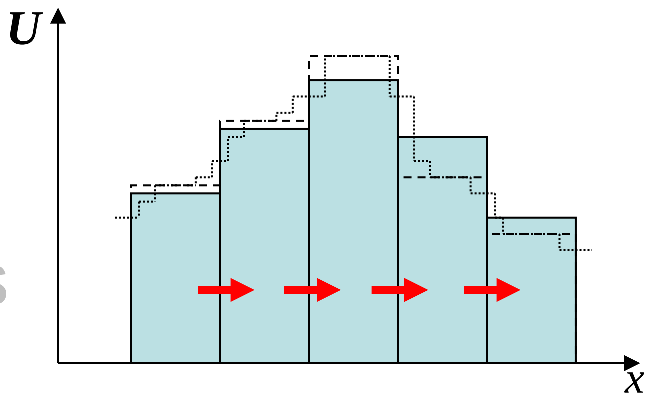
  - Define piecewise constants
  - Solve local Riemann problems
  - Average state variables



Approximate Riemann solver

- Approximate Riemann solver

$$\iint\left(\frac{\partial \boldsymbol{U}}{\partial t}+\frac{\partial \boldsymbol{F}}{\partial x}\right)dxdt=\oint\left(\boldsymbol{U}dx-\boldsymbol{F}dt\right)=0$$



  - Define piecewise constants
  - Solve local Riemann problems
  - Average state variables
  - Derive numerical fluxes from conservation laws

$$\int_{x_i}^{x_{i+1/2}}\boxed{\boldsymbol{U}\left(\frac{x-x_{i+1/2}}{\Delta t};\boldsymbol{U}_i^n,\boldsymbol{U}_{i+1}^n\right)}dx-\left(x_{i+1/2}-x_i\right)\boldsymbol{U}_i^n+\Delta t\left(\underline{\boldsymbol{F}_{i+1/2}}-\boldsymbol{F}_i^n\right)=0$$

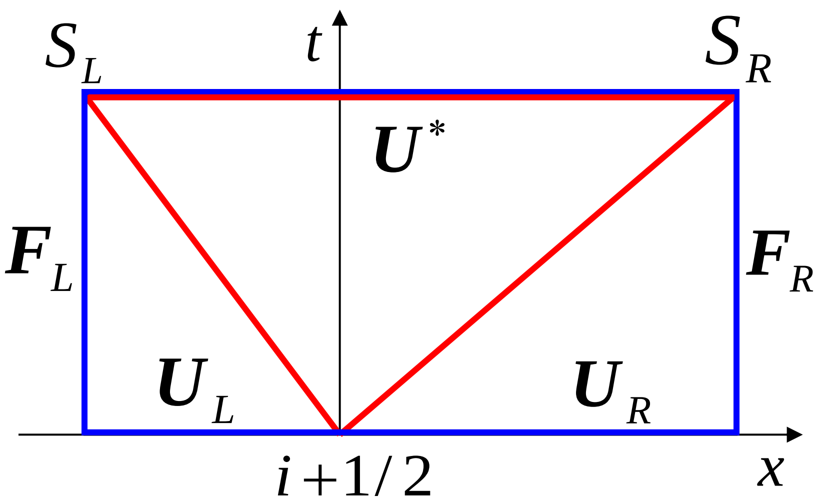
Depend on “quality” of approximate solutions!

Approximate Riemann solver

- Standard approximate Riemann solver
  - Lax-Friedrichs scheme [Lax, 1950's]
  - Godunov scheme [Godunov, 1959]
  - Rusanov scheme [Rusanov, 1961]
  - Roe scheme (HD) [Roe, 1981]
  - HLL scheme [Harten+, 1983]
  - Roe scheme (MHD) [Brio+, 1988]
  - HLLC scheme (HD) [Toro+, 1994; Batten+, 1997]
  - HLLC scheme (MHD) [Gurski, 2004; Li, 2005]
  - HLLD scheme (MHD) [Miyoshi+, 2005]

HLL approximate Riemann solver

- HLL Riemann solver [Harten+, 1983]
  - Conservation laws
  - 2-waves approximation



$S_{R,L}$: max./min. speeds

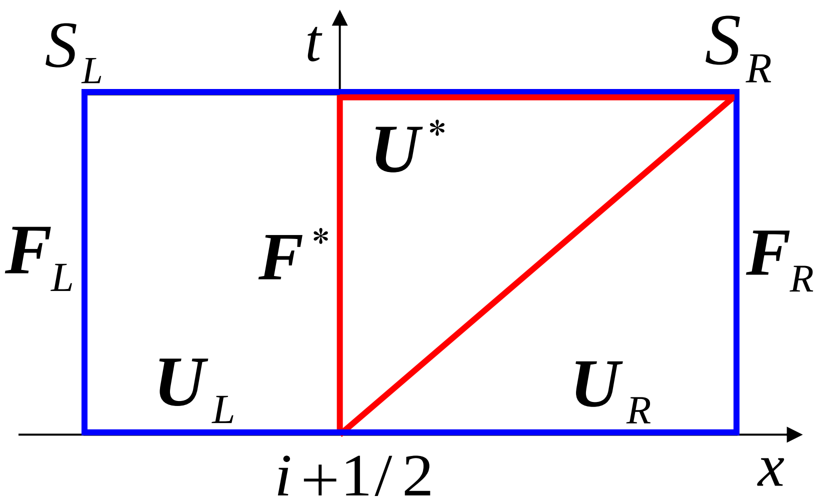
$$S_R = \max\left(u_L + c_L, u_R + c_R, 0\right)$$
$$S_L = \min\left(u_L - c_L, u_R - c_R, 0\right)$$

$$\oint\left(\boldsymbol{U}dx - \boldsymbol{F}dt\right) = 0 \Rightarrow \left(S_R - S_L\right)\boldsymbol{U}^* - S_R\boldsymbol{U}_R + S_L\boldsymbol{U}_L + \boldsymbol{F}_R - \boldsymbol{F}_L = 0$$

- CD/TD/RD cannot be resolved

HLL approximate Riemann solver

- HLL Riemann solver [Harten+, 1983]
  - Conservation laws
  - 2-waves approximation



$S_{R,L}$: max./min. speeds

$$S_R = \max\left(u_L + c_L, u_R + c_R, 0\right)$$

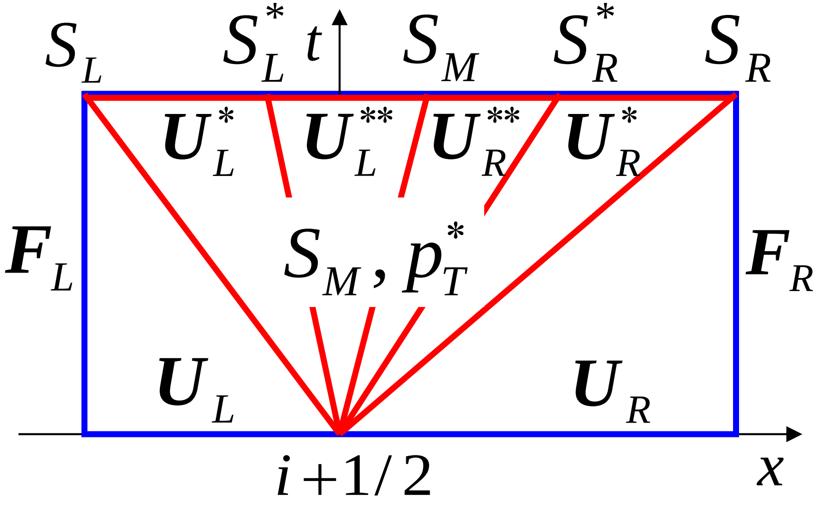
$$S_L = \min\left(u_L - c_L, u_R - c_R, 0\right)$$

$$\oint\left(\boldsymbol{U}dx - \boldsymbol{F}dt\right) = 0 \Rightarrow S_R\boldsymbol{U}^* - S_R\boldsymbol{U}_R + \boldsymbol{F}_R - \boldsymbol{F}^* = 0$$

- CD/TD/RD cannot be resolved

HLLD approximate Riemann solver

- HLLD Riemann solver [Miyoshi+, 2005]
  - Conservation laws
  - 5-waves approximation



$S_{R,L}$: fast magnetosonic wave

S_M : entropy wave

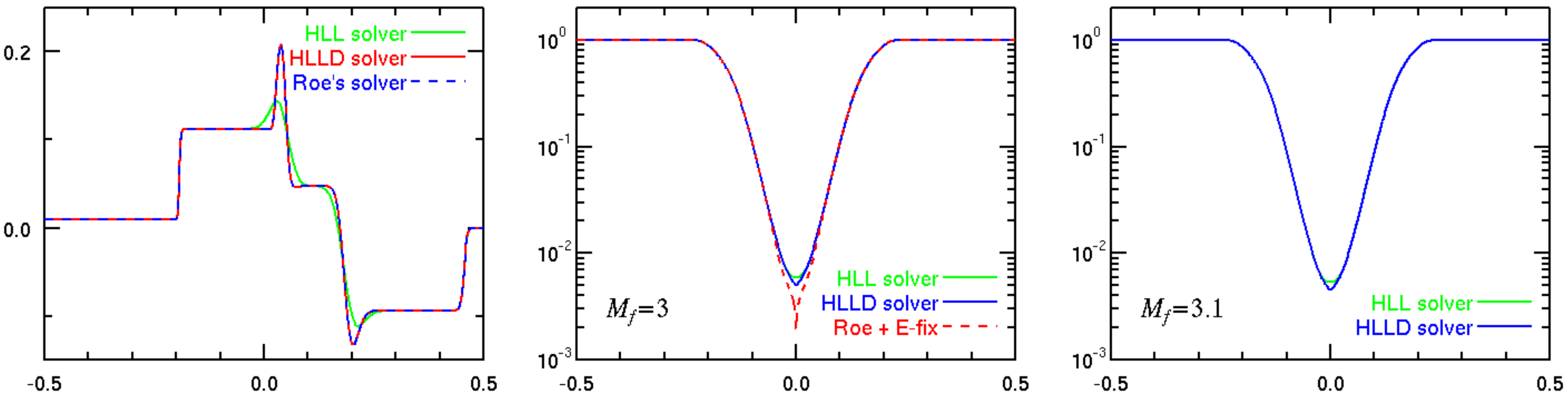
$S_{R,L}^*$: Alfvén wave

$$S_{R,L}\left(\boldsymbol{U}_{R,L}^* - \boldsymbol{U}_{R,L}\right) = \boldsymbol{F}_{R,L}^* - \boldsymbol{F}_{R,L},\ S_{R,L}^*\left(\boldsymbol{U}_{R,L}^{**} - \boldsymbol{U}_{R,L}^*\right) = \boldsymbol{F}_{R,L}^{**} - \boldsymbol{F}_{R,L}^*,$$

$$S_M\left(\boldsymbol{U}_R^{**} - \boldsymbol{U}_L^{**}\right) = \boldsymbol{F}_R^{**} - \boldsymbol{F}_L^{**},\ \frac{1}{\Delta t}\int_{S_L \Delta t}^{S_R \Delta t} \boldsymbol{U}\left(x, t^{n+1}\right) dx + S_R \boldsymbol{U}_R - S_L \boldsymbol{U}_L + \boldsymbol{F}_R - \boldsymbol{F}_L = 0$$

HLLD approximate Riemann solver

- The HLLD Riemann solver
  - is constructed without eigenvectors
  - exactly resolves isolated CD/TD/RD/FS
  - preserves density and pressure positivities
- High-efficiency! High-resolution! Robust!



HLLD approximate Riemann solver

- Established as a standard Riemann solver
  - Comparing numerical methods [Kritsuk+, 2011]

Table 2
Solver Design Specifications for the Eulerian Methods[a]

Name	Base Scheme[b]	Spatial Order[c]	Source Terms[d]	MHD[e]	Time Integration[f]	Directional Splitting[g]
ENZO	FV, HLL	Second	Dedner	Dedner	Second-order RK	Direct
FLASH	FV, HLLD	Second	$\parallel$ Derivative	Third-order CT	Forward Euler	$\perp$ Reconstruction
KT-MHD	FD, CWENO	Third	KT	Third-order CT	Fourth-order RK	Direct
LL-MHD	FV, HLLD	Second	None	Athena CT	Forward Euler	Split
PLUTO	FV, HLLD	Third	Powell	Powell	Fourth-order RK	Direct
PPML	FV, HLLD	Third	None	Athena CT	Forward Euler	$\perp$ Reconstruction
RAMSES	FV, HLLD	Second	None	2D HLLD CT	Forward Euler	$\perp$ Reconstruction
STAGGER	FD, Stagger	Sixth	Tensor	Staggered CT	Third-order Hyman	Direct
ZEUS	FD, van Leer	Second	von Neumann	MOC-CT	Forward Euler	Split

Notes.
[a] See Section 3 and the indicated sections on each topic for more information.
[b] Base method. FD for finite difference, FV for finite volume. FV techniques have the Riemann solver listed, Section 6.3.
[c] Spatial order of accuracy, Section 6.1.
[d] Artificial Viscosity, Section 6.2. "$\parallel$ Derivative" indicates presence of terms proportional to the longitudinal derivative of the magnetic field.
[e] MHD method, Section 6.4.
[f] Time integration method, Section 6.6.3.
[g] Multidimensional technique, Section 6.6.2. "$\perp$ Reconstruction" indicates presence of transverse derivatives in the interface reconstruction.

  - Athena (US), CANS+ (Japan), and many other researches

Challenges

- Challenges to multi-D MHD scheme
  - Comparing numerical methods [Kritsuk+, 2011]

Table 2

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Name	Base Scheme[b]	Spatial Order[c]	Source Terms[d]	MHD[e]	Time Integration[f]	Directional Splitting[g]
ENZO	FV, HLL	Second	Dedner	Dedner	Second-order RK	Direct
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LL-MHD	FV, HLLD	Second	None	Athena CT	Forward Euler	Split
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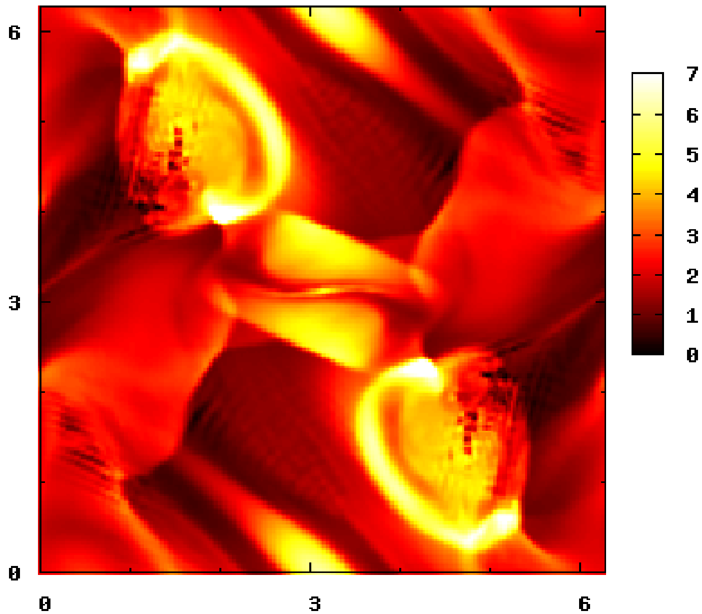
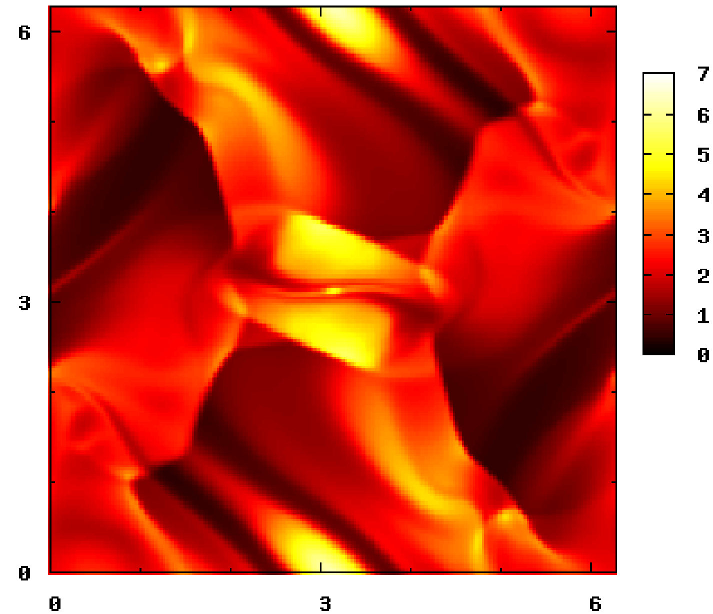
Challenges to multi-D

- Treatment of numerical magnetic monopole



（with correction）



（without correction）

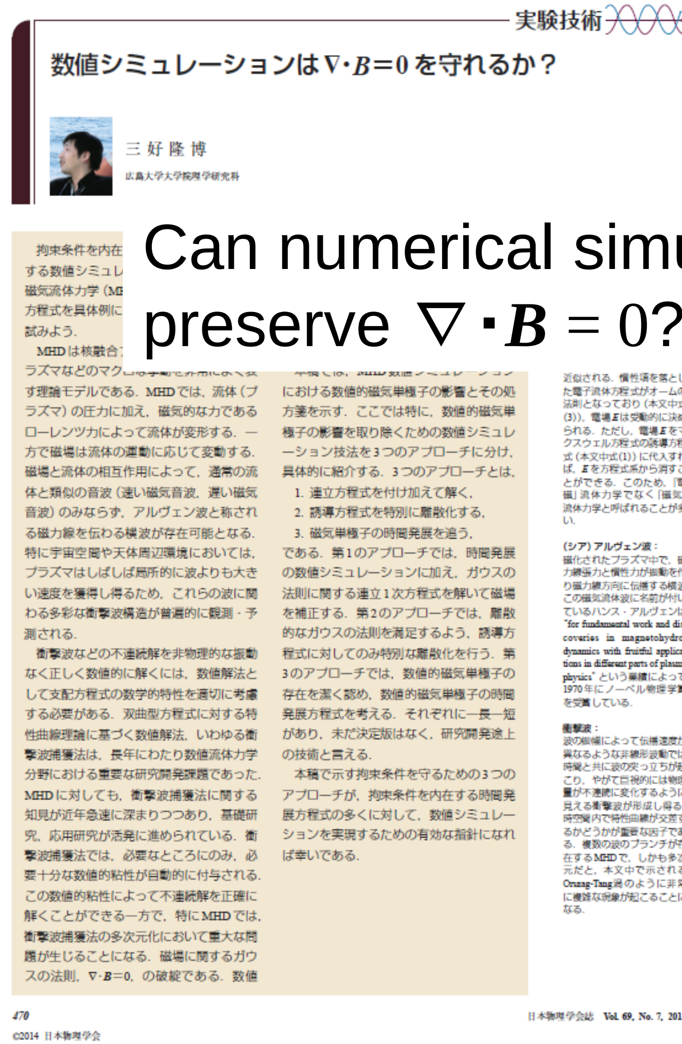
- Negative effect due to unphysical magnetic force

$$-\nabla\cdot\left(B^2/2\,\boldsymbol{I}-\boldsymbol{BB}\right)=\left(\nabla\times\boldsymbol{B}\right)\times\boldsymbol{B}+\boldsymbol{B}\left(\nabla\cdot\boldsymbol{B}\right)$$

- Need divergece-free/divergence-cleaning method!

Challenges to multi-D

- Treatment of numerical magnetic monopole



実験技術

数値シミュレーションは$\nabla\cdot\boldsymbol{B}=0$を守れるか？

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拘束条件を内在
する数値シミュレ
磁気流体力学（MH
方程式を具体例に
試みよう．

MHDは核融合プ
ラズマなどのマク

す理論モデルである．MHDでは，流体（プラズマ）の圧力に加え，磁気的な力であるローレンツ力によって流体が変形する．一方で磁場は流体の運動に応じて変動する．磁場と流体の相互作用によって，通常の流体と類似の音波（速い磁気音波，遅い磁気音波）のみならず，アルヴェン波と称される磁力線を伝わる横波が存在可能となる．特に宇宙空間や天体周辺環境においては，プラズマはしばしば局所的に波よりも大きい速度を獲得し得るため，これらの波に関わる多彩な衝撃波構造が普遍的に観測・予測される．

衝撃波などの不連続解を非物理的な振動なく正しく数値的に解くには，数値解法として支配方程式の数学的特性を適切に考慮する必要がある．双曲型方程式に対する特性曲線理論に基づく数値解法，いわゆる衝撃波捕獲法は，長年にわたり数値流体力学分野における重要な研究開発課題であった．MHDに対しても，衝撃波捕獲法に関する知見が近年急速に深まりつつあり，基礎研究，応用研究が活発に進められている．衝撃波捕獲法では，必要なところにのみ，必要十分な数値的粘性が自動的に付与される．この数値的粘性によって不連続解を正確に解くことができる一方で，特にMHDでは，衝撃波捕獲法の多次元化において重大な問題が生じることになる．磁場に関するガウスの法則，$\nabla\cdot\boldsymbol{B}=0$，の破綻である．数値

における数値的磁気単極子の影響とその処方箋を示す．ここでは特に，数値的磁気単極子の影響を取り除くための数値シミュレーション技法を3つのアプローチに分け，具体的に紹介する．3つのアプローチとは，

1. 連立方程式を付け加えて解く，
2. 誘導方程式を特別に離散化する，
3. 磁気単極子の時間発展を追う，

である．第1のアプローチでは，時間発展の数値シミュレーションに加え，ガウスの法則に関する連立1次方程式を解いて磁場を補正する．第2のアプローチでは，離散的なガウスの法則を満足するよう，誘導方程式に対してのみ特別な離散化を行う．第3のアプローチでは，数値的磁気単極子の存在を潔く認め，数値的磁気単極子の時間発展方程式を考える．それぞれに一長一短があり，未だ決定版はなく，研究開発途上の技術と言える．

本稿で示す拘束条件を守るための3つのアプローチが，拘束条件を内在する時間発展方程式の多くに対して，数値シミュレーションを実現するための有効な指針になれば幸いである．

近似される．慣性項を落とした電子流体方程式がオームの法則となっており（本文中式(3)），電場$\boldsymbol{E}$は受動的に決められる．ただし，電場$\boldsymbol{E}$をマクスウェル方程式の誘導方程式（本文中式(1)）に代入すれば，$\boldsymbol{E}$を方程式系から消すことができる．このため，「電磁」流体力学でなく「磁気」流体力学と呼ばれることが多い．

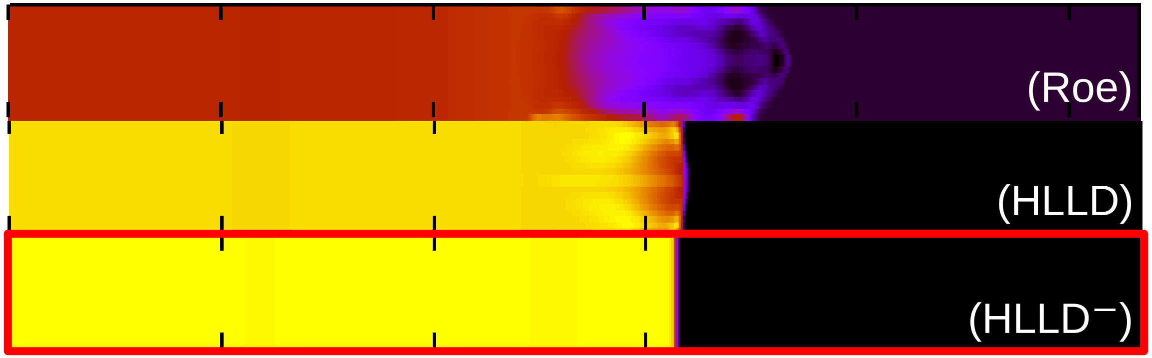
(シア) アルヴェン波：
磁化されたプラズマ中で，磁力線張力と慣性力が振動を作り磁力線方向に伝播する横波．この磁気流体波に名前が付いているハンス・アルヴェンは，"for fundamental work and discoveries in magnetohydrodynamics with fruitful applications in different parts of plasma physics" という業績によって1970年にノーベル物理学賞を受賞している．

衝撃波：
波の振幅によって伝播速度が異なるような非線形波動では，時間と共に波の突っ立ちが起こり，やがて巨視的には物理量が不連続に変化するように見える衝撃波が形成し得る．時空間内で特性曲線が交差するかどうかが重要な因子である．複数の波のブランチが存在するMHDで，しかも多次元だと，本文中で示されるOrszag-Tang渦のように非常に複雑な現象が起こることになる．

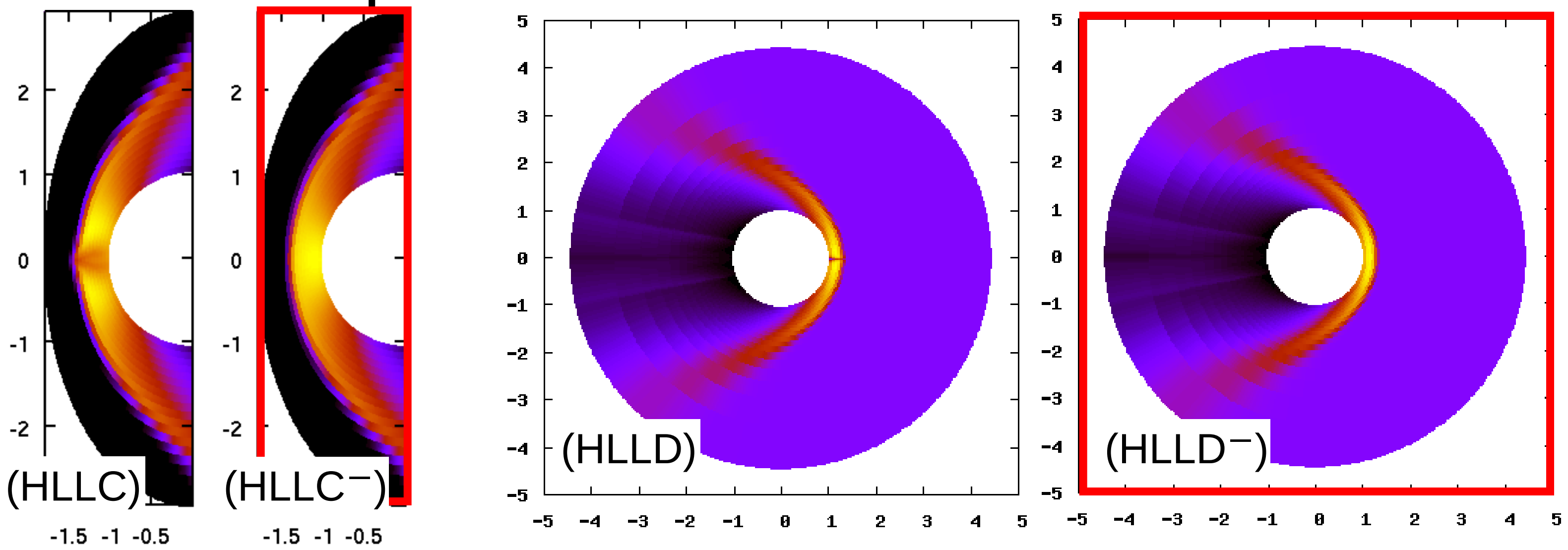
Can numerical simulations preserve $\nabla\cdot\boldsymbol{B}=0$?

Challenges to multi-D

- Numerical shock instabilities
  - Odd-even decoupling



  - Carbuncle phenomena



Challenges

- Challenges to higher-order MHD scheme
  - Comparing numerical methods [Kritsuk+, 2011]

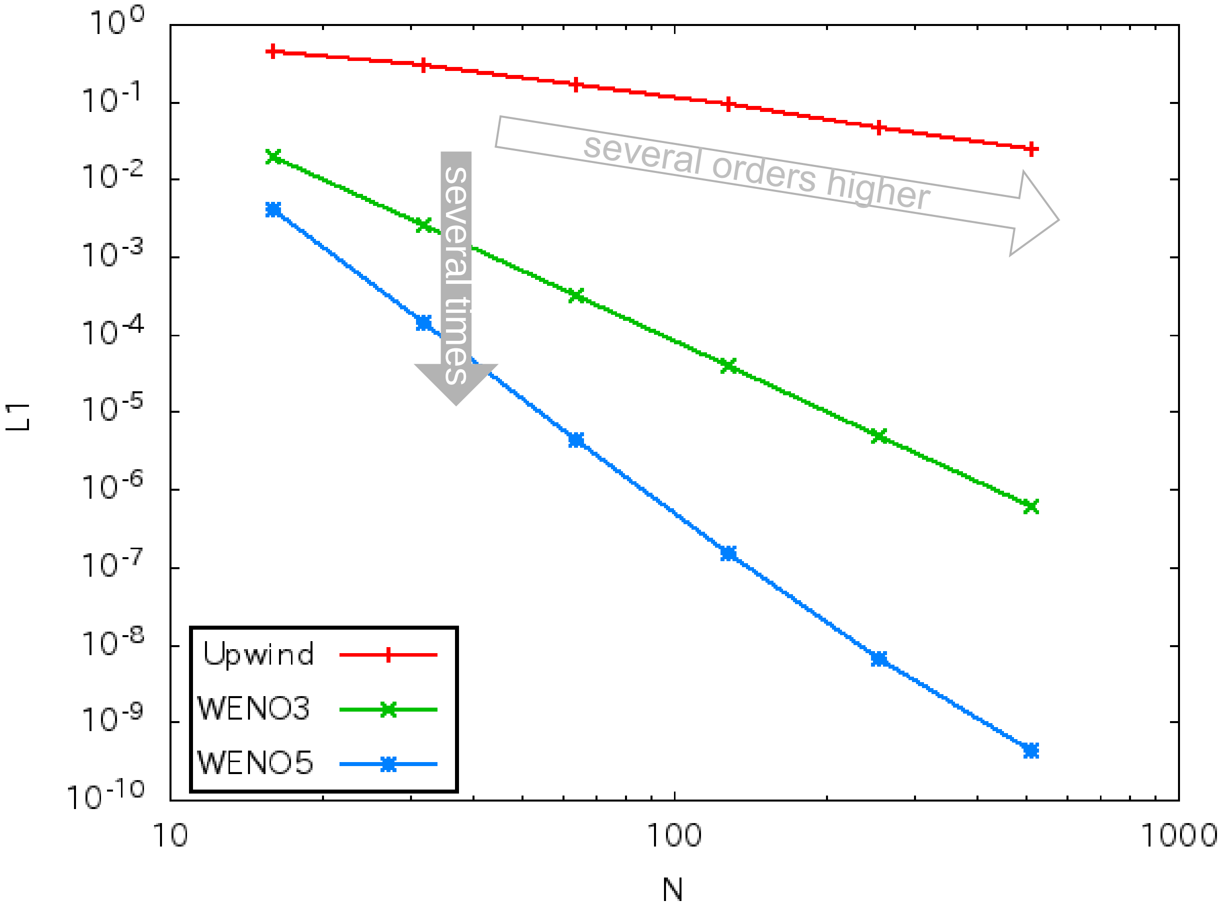
Table 2
Solver Design Specifications for the Eulerian Methods[a]

Name	Base Scheme[b]	Spatial Order[c]	Source Terms[d]	MHD[e]	Time Integration[f]	Directional Splitting[g]
ENZO	FV, HLL	Second	Dedner	Dedner	Second-order RK	Direct
FLASH	FV, HLLD	Second	∥ Derivative	Third-order CT	Forward Euler	⊥ Reconstruction
KT-MHD	FD, CWENO	Third	KT	Third-order CT	Fourth-order RK	Direct
LL-MHD	FV, HLLD	Second	None	Athena CT	Forward Euler	Split
PLUTO	FV, HLLD	Third	Powell	Powell	Fourth-order RK	Direct
PPML	FV, HLLD	Third	None	Athena CT	Forward Euler	⊥ Reconstruction
RAMSES	FV, HLLD	Second	None	2D HLLD CT	Forward Euler	⊥ Reconstruction
STAGGER	FD, Stagger	Sixth	Tensor	Staggered CT	Third-order Hyman	Direct
ZEUS	FD, van Leer	Second	von Neumann	MOC-CT	Forward Euler	Split

Notes.
[a] See Section 3 and the indicated sections on each topic for more information.
[b] Base method. FD for finite difference, FV for finite volume. FV techniques have the Riemann solver listed, Section 6.3.
[c] Spatial order of accuracy, Section 6.1.
[d] Artificial Viscosity, Section 6.2. "∥ Derivative" indicates presence of terms proportional to the longitudinal derivative of the magnetic field.
[e] MHD method, Section 6.4.
[f] Time integration method, Section 6.6.3.
[g] Multidimensional technique, Section 6.6.2. "⊥ Reconstruction" indicates presence of transverse derivatives in the interface reconstruction.

Challenges to higher-order

- Importance of higher-order methods
  - Error of nth-order method vs. Computational cost



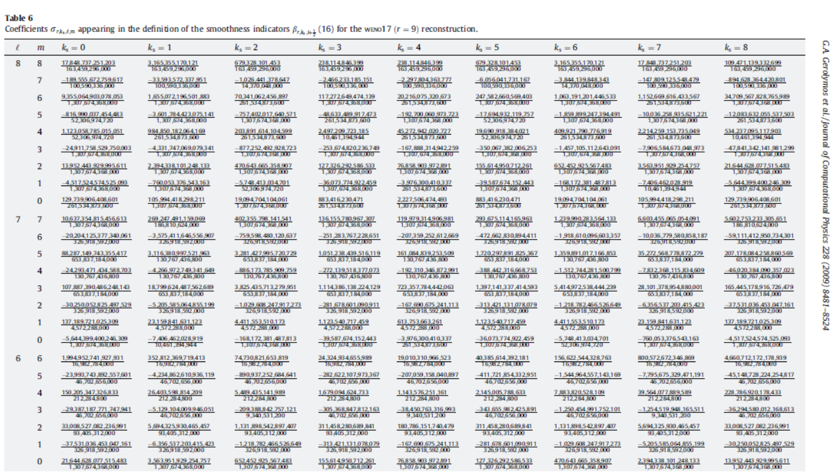
$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

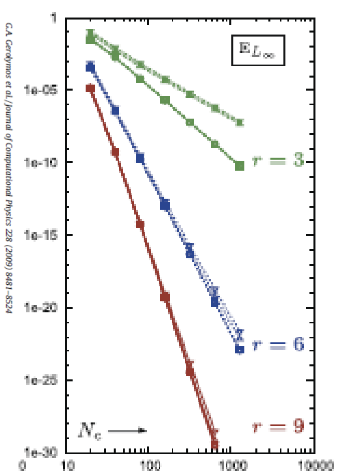
$$u(x, t=0) = \sin(2\pi x)$$

$$L1 \equiv \frac{1}{N}\sum_i \left|u_i - u(x_i, t)\right|$$

Challenges to higher-order

- Godunov's theorem
  - Any *linear* monotone scheme (non-oscillatory scheme) can be at most first-order accurate
  - This statement suggests that higher-order non-oscillatory scheme can be constructed as a nonlinear scheme
    - TVD, MUSCL, PPM, WENO, etc.





Very-high-order WENO (up to 17th-order) [Gerolymos+, 2009]

Challenges

- *Multi-dimensional higher-order divergence-free* scheme is one of the goals of shock capturing scheme for MHD

Summary

- I have reported current status and challenges of robust shock capturing schemes for MHD
  - The HLLD has been established as a standard MHD solver in the field of astrophysics
  - Multi-D shock capturing scheme for MHD is one of the challenges
    - Treatment of numerical magnetic monopole
    - Treatment of numerical shock instabilities
  - Higher-order shock capturing scheme for MHD is one of the challenges
  - Study on shock capturing scheme for two-fluid / extended MHD is now progressing…