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On the treatment of polarization drift in electromagnetic nonlinear gyrokinetic equations

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Motivation

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- While some give impressions (and actually believe) their GK codes can solve everything from hyper fine-scale turbulence up to MHD modes at a fraction of system size, nonlinear gyrokinetic equations don't recover all the terms in drift-kinetic equations.

[Brizard and Hahm, RMP 2007][Kulsrud, Basic Plasma Physics 1983]

- The standard gyrokinetic equations contain no polarization drift, but a polarization correction to the gyrocenter density in the Poisson equation. Due to this, there are theoretical issues concerning nonlinear terms involving polarization drift which haven't been explored in the context of gyrokinetic theory.

[Hahm PoP 1996][Wang and Hahm, PoP 2010]

- One can introduce polarization drift to the gyrokinetic Vlasov equation, but that leads to some subtleties which are not fully appreciated by the community and some confusion.

[Scott and Miyamoto, JPSJ 2009]
[Comment by S. Leerink et al. to and response by Wang and Hahm, PoP 2010]

Motivation (2)

- In (drift-kinetic) studies on the nonlinear saturation of Toroidal Alfvén Eigenmodes in particular, there are terms linked to a ponderomotive force associated with polarization drift which aren't made explicit in the standard gyrokinetic equations

$$\widehat{\mathbf{b}} \cdot \delta\mathbf{B}_{\perp} \times \delta\mathbf{J}_{\perp pol}$$

[Hahm and Chen, PRL 1995]

- Formulations including polarization drift terms in the electrostatic case have been derived, but do not contain these electromagnetic terms.

[Wang and Hahm, PoP 2010]

- Including polarization drift in the dynamic equations may facilitate analytic applications (e.g., residual stress calculation)

[McDevitt et al., PoP 2009]

→ Objective: derive consistent gyrokinetic equations containing polarization drift with magnetic perturbations.

Other Polarization drift methods

- There exist different ways of introducing polarization drift in gyrokinetic theory.

- For instance, an inertial term responsible for the compressional Alfvén wave has been recovered through polarization drift associated with an induction electric field $-c^{-1}\frac{\partial}{\partial t}\delta\mathbf{A}_\perp$ in the context of a high-frequency gyrokinetic approach.

[Qin and Tang, PoP 2004]

- In another example, addressing electrostatic fluctuations in a linear uniform background magnetic field, the polarization drift appears in the difference between guiding-center and gyrocenter positions.

[Brizard and Mishchenko, PoP 2009]

- In both works, they achieve this by including a ω/Ω correction term in S_1 at the first order.

Lie-Transform Methods

What are Lie Transforms?

- Lie transforms are perturbative transformations taking advantage of a scale separation in the system, which convert both phase-space (coordinates) and dynamics (Lagrangian) self-consistently.
- Their most attractive property is that they preserve the Hamiltonian flow (Liouville theorem), so that conservation laws still apply to the system.
- The method itself is coordinate-independent, so canonical coordinates are not required.

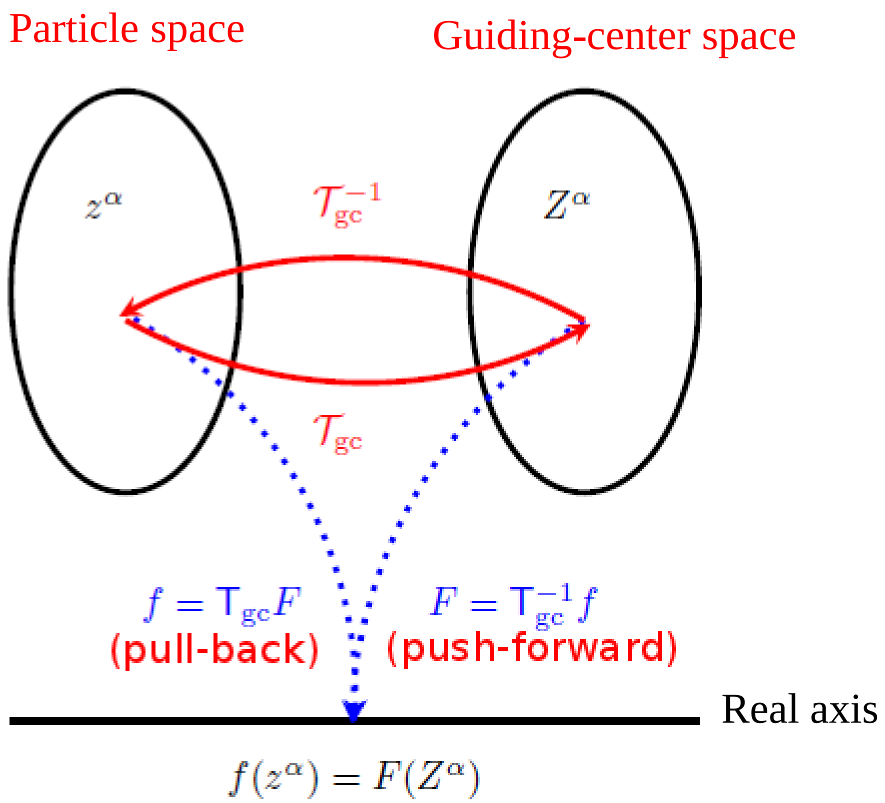
$$\mathsf{T}_{gc} A = e^{i\mathcal{L}} A$$

- The fast time scales in the dynamics are eliminated at each order in the transformation. This gives us a generating vector field at each order, with which we can deduce the transformed coordinates and operators.

$$\begin{aligned}\mathcal{T}_{gc} z^a &= z^a + \epsilon G_1^a + \ldots \\ \mathsf{T}_{gc} A &= A + \epsilon G_1^b \frac{\partial A}{\partial z^b} + \ldots\end{aligned}$$

[Littlejohn, PF 1981][Cary and Littlejohn, AoP 1983][Brizard and Hahm, RMP 2007]

Guiding-center transformation



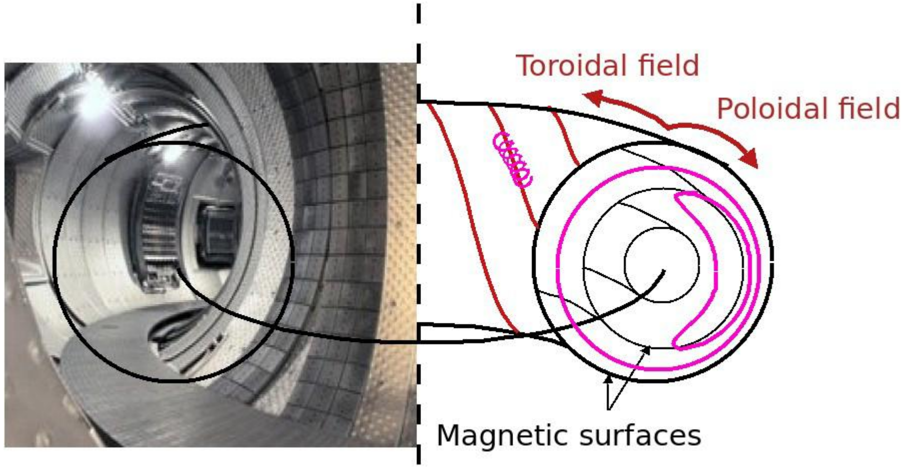
- Particle and guiding-center distribution functions describe the same physics
- All these expressions are equivalent:

$$F(\mathbf{Z}) = \mathsf{T}_{gc}^{-1} f(\mathbf{Z}) = F(\mathcal{T}_{gc}\mathbf{z}) = f(\mathbf{z})$$

$$f(\mathbf{z}) = \mathsf{T}_{gc} F(\mathbf{z}) = f(\mathcal{T}_{gc}^{-1}\mathbf{Z}) = F(\mathbf{Z})$$

Gyrocenter Dynamics

Gyrokinetic Premise and Orderings



- Adiabatic invariant associated with the fast gyration motion: magnetic moment

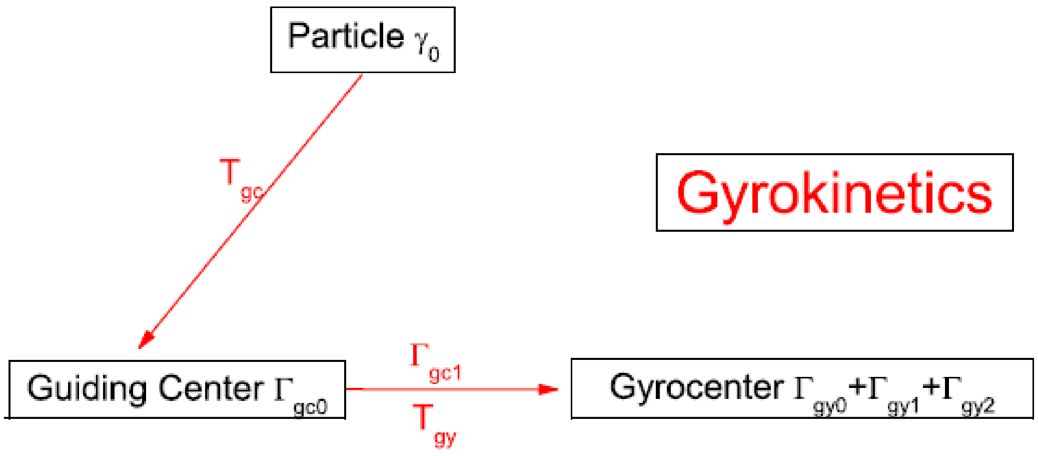
$$\mu \equiv m v_\perp^2 / 2B$$

- Orderings used in this work:

$$\rho/L \sim \omega/\Omega \sim e\delta\phi/T \sim \delta B/B \sim k_\parallel \rho_i \sim \epsilon \ll 1$$

$$k_\perp \rho_i \sim 1$$

Phase-Space Transformations



- Perturbative transformations which eliminate fast time scales from the dynamics at each order.
- Small parameters:

ρ/L (guiding-center)

$e\delta\Phi/T, \delta B/B$ (gyrocenter)

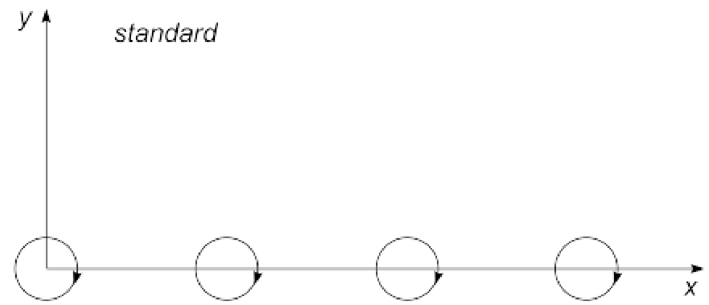
[Brizard and Hahm, RMP 2007]

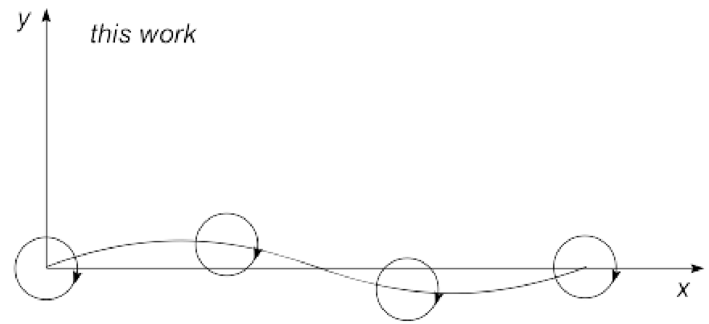
Gyrocenter Lagrangian

- We include the gyro-averaged perturbed ExB drift velocity and magnetic perturbations explicitely in the gyrocenter Lagrangian:

$$\langle \delta \mathbf{u}_{Egc} \rangle \equiv c\widehat{\mathbf{b}} \times \nabla \langle \delta\phi_{gc} \rangle / B$$

- This amounts to a new reference frame which follows the particle orbit more closely along the potential fluctuations, especially in high ExB-shear areas (e.g. transport barriers).





[Wang and Hahm, PoP 2010]

$$\Gamma_{gc} = \left[\frac{e}{c}\mathbf{A} + \epsilon_\delta \delta A_{\|gc} \widehat{\mathbf{b}} + mU\widehat{\mathbf{b}} \right] \cdot \mathrm{d}\mathbf{R} + \frac{mc}{e}\mu \mathrm{d}\varphi - \left(\frac{1}{2}mU^2 + \mu B + \epsilon_\delta e\delta\phi_{gc} \right) \mathrm{d}t$$

$$\Gamma_{gy} = \left[\frac{e}{c}\mathbf{A} + \epsilon_\delta \langle \delta A_{\|gc} \rangle \widehat{\mathbf{b}} + mU_{gy}\widehat{\mathbf{b}} + \epsilon_\delta \langle \delta \mathbf{u}_{Egc} \rangle \right] \cdot \mathrm{d}\mathbf{R}_{gy} + \frac{mc}{e}\mu_{gy} \mathrm{d}\varphi_{gy} - \left(\frac{1}{2}mU_{gy}^2 + \mu_{gy} B + \epsilon_\delta e\delta\Psi_{gy} \right) \mathrm{d}t$$

Comparison with the Standard Gyrokinetic Formalism

- The gyrocenter position is redefined with respect to the standard gyrokinetic expressions:

	Guiding center	Gyrocenter
Standard gyrokinetics	$\mathbf{X} = \mathbf{x} - \boldsymbol{\rho}_0$	$\mathbf{R} = \mathbf{x} - \boldsymbol{\rho}_0 - \left(mB^*_{0\|}\right)^{-1} \left(\frac{\partial S_1}{\partial U}\mathbf{B}^*_0 + \frac{mc}{e}\widehat{\mathbf{b}} \times \nabla S_1\right)$
This work	$\mathbf{X} = \mathbf{x} - \boldsymbol{\rho}_0$	$\mathbf{R} = \mathbf{x} - \boldsymbol{\rho}_0 - \left(mB^*_{0\|}\right)^{-1} \left(\frac{\partial S_1}{\partial U}\mathbf{B}^*_0 + \frac{mc}{e}\widehat{\mathbf{b}} \times \nabla S_1\right) + \left(mc/eB^*_{0\|}\right)\widehat{\mathbf{b}} \times \langle \delta \mathbf{u}_{\mathrm{Egc}} \rangle$

- Note that magnetic perturbations are included in the gyrocenter gauge function,

$$S_1 \equiv \frac{e}{\Omega} \int d\varphi \left(\delta\phi_{\mathrm{gc}} - \frac{U}{c}\delta A_{\|\mathrm{gc}} - \left\langle \delta\phi_{\mathrm{gc}} - \frac{U}{c}\delta A_{\|\mathrm{gc}} \right\rangle\right)$$

Gyrocenter Hamiltonian

- Recall the guiding-center Hamiltonian:

$$H = \frac{1}{2}mU^2 + \mu B(\mathbf{R}) + e\delta\phi_{gc}$$

- The gyrocenter Hamiltonian has the form

$$H_{gy} = \frac{1}{2}mU_{gy}^2 + \mu_{gy}B(\mathbf{R}_{gy}) + e\delta\Psi_{gy}$$

- The effective gyrocenter potential is expressed in terms of gyro-averages of the perturbed potentials:

$$e\delta\Psi_{gy} = e\left\langle\delta\phi_{gc}\right\rangle - \frac{e^2}{2mc^2}\left\langle\widetilde{\delta A}_{\|gc}^2\right\rangle - \frac{e^2}{2B}\frac{\partial}{\partial\mu}\left\langle\left(\widetilde{\delta\Psi}\right)^2\right\rangle + \frac{mB}{B_{0\|}^*}\left|\left\langle\delta\mathbf{u}_{Egc}\right\rangle\right|^2$$

$$\widetilde{\delta\Psi} \equiv \widetilde{\delta\phi}_{gc} - \frac{U}{c}\widetilde{\delta A}_{\|gc}$$

$$\widetilde{\delta\phi}_{gc} \equiv \delta\phi_{gc} - \left\langle\delta\phi_{gc}\right\rangle, \quad \widetilde{\delta A}_{\|gc} \equiv \delta A_{\|gc} - \left\langle\delta A_{\|gc}\right\rangle$$

Euler-Lagrange Equations

- From the Euler-Lagrange equations, there are additional ExB contributions:

$$\frac{d\mathbf{R}}{dt} = \frac{1}{B_{\parallel}^{*}} \left[\widehat{\mathbf{b}} \times \left(\frac{\mu}{m\Omega} \nabla B + c \nabla \delta\Psi_{gy} \right) + \left(U + \frac{e}{m} \frac{\partial \delta\Psi_{gy}}{\partial U} \right) \mathbf{B}^{*} - \frac{c}{\Omega} \frac{\partial}{\partial t} \nabla_{\perp} \left\langle \delta\phi_{gc} \right\rangle \right]$$

$$\frac{dU}{dt} = -\frac{1}{mB_{\parallel}^{*}} \mathbf{B}^{*} \cdot \left(\mu \nabla B + e \nabla \delta\Psi_{gy} + m \frac{\partial}{\partial t} \left\langle \delta\mathbf{u}_{Egc} \right\rangle \right)$$

- The resulting Vlasov equation for the gyrocenter distribution becomes

$$\frac{\partial F_{gy}}{\partial t} + \frac{d\mathbf{R}}{dt} \cdot \nabla F_{gy} + \frac{dU}{dt} \frac{\partial F_{gy}}{\partial U} = 0$$

(recall $d\mu/dt = 0$ and the gyrocenter distribution is gyrophase-independent)

Euler-Lagrange Equations (2)

- The equations of motion involve the modified magnetic field and phase-space volume with magnetic perturbation and polarization terms,

$$\mathbf{B}^* \equiv \mathbf{B} + \frac{mcU}{e}\nabla \times \widehat{\mathbf{b}} + \frac{mc}{e}\nabla \times \langle \delta \mathbf{u}_{Egc} \rangle + \langle \delta \mathbf{B}_{gy} \rangle$$

$$B_{\parallel}^* \equiv B + \frac{mcU}{e}\widehat{\mathbf{b}} \cdot \nabla \times \widehat{\mathbf{b}} + \frac{mc^2}{e}\nabla_{\perp} \cdot \left(\frac{1}{B}\nabla_{\perp} \langle \delta \phi_{gc} \rangle \right)$$

- The formulation we used shows explicit second-order terms corresponding to the polarization drift

$$\mathbf{u}_{pol} \equiv -\frac{c}{\Omega B}\frac{\partial}{\partial t}\nabla_{\perp} \langle \delta \phi_{gc} \rangle$$

and its associated nonlinear ponderomotive force which includes a magnetic term found in drift-kinetic theory,

$$-\frac{1}{B}\delta \mathbf{B}_{\perp} \cdot \frac{\partial}{\partial t}\delta \mathbf{u}_E = \frac{e}{mc}\widehat{\mathbf{b}} \cdot \delta \mathbf{B}_{\perp} \times \mathbf{u}_{pol}$$

Energy invariant

- The global energy invariant is expressed in the following manner, ignoring FLR effects for electrons:

$$E = \int d^3\mathbf{x} \overbrace{\frac{1}{8\pi}\left(\left|\nabla\delta\phi\right|^2 + \left|\mathbf{B}+\delta\mathbf{B}\right|^2\right)}^{\text{Electromagnetic field energy}} + \int d^6\mathbf{z} \overbrace{\frac{1}{2}m_e v^2}^{\text{Electron energy}} f_e$$

$$+ \int \frac{B_\parallel^*}{m} d^3\mathbf{R}\, dU\, d\mu F_i\left(Z\right)\left[\underbrace{\frac{1}{2}mU^2 + \mu B}_{\text{Gyrocenter kinetic energy}} + \underbrace{e\Psi_{gy} - e\left\langle \mathsf{T}_{gy}^{-1}\delta\phi_{gc}\right\rangle}_{\text{Effective electromagnetic potential energy}}\right]$$

- The effective electromagnetic potential energy is

$$e\Psi_{gy} - e\left\langle \mathsf{T}_{gy}^{-1}\delta\phi_{gc}\right\rangle = \frac{1}{2}\frac{e^2}{mc^2}\left\langle\left(\widetilde{\delta A_{\parallel gc}}\right)^2\right\rangle - \frac{e^2}{2B}\frac{\partial}{\partial\mu}\left\langle\left(\widetilde{\delta\Psi}\right)^2\right\rangle + \frac{e^2}{B}\frac{\partial}{\partial\mu}\left\langle\widetilde{\delta\Psi}\widetilde{\delta\phi}_{gc}\right\rangle$$

[Sugama, PoP 2000][Brizard, PoP 2000]

Poisson and Ampère Equations

- The perturbed Poisson-Ampère equations on the electromagnetic potentials are:

$$\nabla^2 \delta\phi = -4\pi \sum_s q_s n_s(\mathbf{x})$$

$$\nabla_\perp^2 \delta A_\| = -\frac{4\pi}{c} \sum_s j_{\|s}(\mathbf{x})$$

- These are calculated in local space, but must be determined from the gyrocenter distribution function(s).

$$\nabla^2 \delta\phi = -4\pi \sum_s q_s \int \mathcal{J} d^6 Z \left\langle \mathsf{T}_{gy}^{-1} \left(\delta^3 \left(\mathbf{R} + \boldsymbol{\rho}_0 - \mathbf{x} \right) \right) \right\rangle F_{gy,s}$$

$$\nabla_\perp^2 \delta A_\| = -\frac{4\pi}{c} \sum_s q_s \int \mathcal{J} d^6 Z \left\langle \mathsf{T}_{gy}^{-1} \left(U \delta^3 \left(\mathbf{R} + \boldsymbol{\rho}_0 - \mathbf{x} \right) \right) \right\rangle F_{gy,s}$$

- The resulting moments are <u>not</u> simply the zeroth and first-order moments of the gyrocenter distribution function! There are correction ("shielding") terms corresponding to the discrepancy between particle and gyrocenter positions.

Polarization Density and Magnetization Current

- Total density is the sum of the gyrocenter density (zeroth-moment of the gyrocenter distribution function F_{gy}) and the corrections arising from the gyro-center transformation ("polarization density")

$$n = N_{gy} + n_{pol}$$

$$N_{gy} = \int \mathcal{J} d^6 Z F_{gy}$$

[Wang and Hahm, PoP 2010]

- Total current density is the sum of the gyrocenter current density (first moment of the gyrocenter distribution function F_{gy}) and the corrections arising from the gyrocenter transformation ("magnetization current")

$$j_{\|} = j_{gy} + j_{\| mag}$$

$$j_{gy} = \int \mathcal{J} d^6 Z U F_{gy}$$

- Note that our definition of the Jacobian $\mathcal{J} = B_{\|}^{*}$ is different from the standard approach and will warrant a second-order correction to the standard polarization density and magnetization current.

Polarization Density and Magnetization Current (2)

- The ion density and parallel current are deduced from the expressions for the gyrocenter generating vector field, with unperturbed parts $\overline{N}_{\mathrm{i}}$ and $\overline{J}_{\|\mathrm{i}}$:

$$\begin{aligned} n_{\mathrm{i}} &= \int \frac{B^*_{\|0}}{m} d^3\mathbf{R}\, d\mu\, dU\, d\varphi\, F_{\mathrm{i}} \left\langle \left[1 - \left(\mathbf{G}_1^{\mathbf{R}} \cdot \nabla + G_1^{\mu} \frac{\partial}{\partial \mu} + G_1^{\varphi} \frac{\partial}{\partial \varphi} \right) \right.\right. \\ &\left.\left. + \frac{mc^2}{eB^*_{\|0}} \nabla_\perp \cdot \left(\frac{1}{B} \nabla_\perp \langle \delta\phi_{\mathrm{gc}} \rangle \right) \right] \delta^3 \left(\mathbf{R} + \boldsymbol{\rho}_0 - \mathbf{x} \right) \right\rangle \\ j_{\|\mathrm{i}} &= e \int \frac{B^*_{\|0}}{m} d^3\mathbf{R}\, d\mu\, dU\, d\varphi\, F_{\mathrm{i}} \left\langle \left[U - \left(G_1^U + U \mathbf{G}_1^{\mathbf{R}} \cdot \nabla + U G_1^{\mu} \frac{\partial}{\partial \mu} + U G_1^{\varphi} \frac{\partial}{\partial \varphi} \right) \right.\right. \\ &\left.\left. + U \frac{mc^2}{eB^*_{\|0}} \nabla_\perp \cdot \left(\frac{1}{B} \nabla_\perp \langle \delta\phi_{\mathrm{gc}} \rangle \right) \right] \delta^3 \left(\mathbf{R} + \boldsymbol{\rho}_0 - \mathbf{x} \right) \right\rangle \end{aligned}$$

- Here, $B^*_{\|0}$ is the guiding-center (unperturbed) Jacobian and the last terms correspond to the contribution of the perturbed part of the gyrocenter Jacobian.

Limiting Cases

Drift-Kinetic Limit: Parallel Dynamics

- The long-wavelength form $k_\perp \rho_i \ll 1$ of the effective gyrocenter potential is

$$e\delta\Psi_{\mathrm{gy}} = e\left\langle \delta\phi_{\mathrm{gc}} \right\rangle + \underbrace{\frac{1}{2} m \left|\delta\mathbf{u}_{\mathrm{E}}\right|^2}_{\text{Electrostatic ExB term}} + \underbrace{\frac{1}{2}\left(\mu B - mU^2\right)\left|\frac{\delta\mathbf{B}_\perp}{B}\right|^2}_{\text{Pressure anisotropy term linked to transit time magnetic pumping}} - \underbrace{mU\,\delta\mathbf{u}_{\mathrm{E}} \cdot \frac{\delta\mathbf{B}_\perp}{B}}_{\text{Parallel perturbed ExB drift}}$$

- Ignoring third-order terms, the parallel acceleration becomes very similar to the drift-kinetic expression:

$$\frac{dU}{dt} = -\frac{1}{m}\mathbf{b}_{\mathrm{tot}} \cdot \left[m\frac{D}{Dt}\delta\mathbf{u}_{\mathrm{E}} + \mu\nabla B_{\mathrm{tot}} - e\delta\mathbf{E} \right]$$

with a convective derivative of the perturbed ExB drift

$$D/Dt \equiv \partial/\partial t + \left[U\widehat{\mathbf{b}} + \delta\mathbf{u}_{\mathrm{E}}\right]\cdot\nabla$$

and the perturbed electric field

$$\delta\mathbf{E} = -\left(\nabla\delta\phi + \widehat{\mathbf{b}}\,\partial\delta A_{\|}/\partial t\right)$$

This has not been demonstrated before using conventional nonlinear gyrokinetics!

Drift-kinetic Limit: Polarization Density and Magnetization Current

- The expressions for the ion density and parallel current include polarization and magnetization effects:

$$
\begin{aligned}
n_{\mathrm{i}} &= \overline{N}_{\mathrm{i}} + \nabla\cdot\int \frac{B_{\parallel}^{*}}{m} dU\,d\mu\,d\varphi F_{\mathrm{i}} \left\langle \mathbf{G}_1^{\mathbf{R}} + G_1^{\mu}\frac{\partial\boldsymbol{\rho}_0}{\partial\mu} + G_1^{\varphi}\frac{\partial\boldsymbol{\rho}_0}{\partial\varphi} \right\rangle \\
&+ \int dU\,d\mu\,d\varphi F_{\mathrm{i}} \frac{c^2}{e}\nabla_{\perp}\cdot\left(\frac{1}{B}\nabla_{\perp}\left\langle\delta\phi_{\mathrm{gc}}\right\rangle\right)
\end{aligned}
$$

$$
\begin{aligned}
j_{\parallel\mathrm{i}} &= \overline{J}_{\parallel\mathrm{i}} + \nabla\cdot\int \frac{B_{\parallel}^{*}}{m} dU\,d\mu\,d\varphi F_{\mathrm{i}} \left\langle G_1^{U}\boldsymbol{\rho}_0 + U\left(\mathbf{G}_1^{\mathbf{R}} + G_1^{\mu}\frac{\partial\boldsymbol{\rho}_0}{\partial\mu} + G_1^{\varphi}\frac{\partial\boldsymbol{\rho}_0}{\partial\varphi}\right) \right\rangle \\
&+ \int dU\,d\mu\,d\varphi F_{\mathrm{i}} \left[\frac{B_{\parallel 0}^{*}}{m}\left\langle G_1^{U}\right\rangle + U\frac{c^2}{e}\nabla_{\perp}\cdot\left(\frac{1}{B}\nabla_{\perp}\left\langle\delta\phi_{\mathrm{gc}}\right\rangle\right)\right]
\end{aligned}
$$

- In particular, the divergence term leads to the magnetic component of the gyrocenter polarization vector and magnetization vector respectively. The electric component cancels out in this formalism.

$$
\left\langle \mathbf{G}_1^{\mathbf{R}} + G_1^{\mu}\frac{\partial\boldsymbol{\rho}_0}{\partial\mu} + G_1^{\varphi}\frac{\partial\boldsymbol{\rho}_0}{\partial\varphi} \right\rangle \simeq -\frac{mc^2}{eB^2}\nabla_{\perp}\delta\phi + \frac{mc^2}{eB^2}\nabla_{\perp}\left(\delta\phi - \frac{U}{c}\delta A_{\parallel}\right) \simeq -U\frac{mc}{eB^2}\nabla_{\perp}\delta A_{\parallel}
$$

$$
\left\langle G_1^{U}\boldsymbol{\rho}_0\right\rangle \simeq \frac{c\mu}{eB}\nabla_{\perp}\delta A_{\parallel}
$$

Drift-Kinetic Limit: Poisson-Ampère Equations

- The Poisson-Ampère equations are deduced from the general expressions in the long-wavelength limit.

$$
\begin{aligned}
\nabla^2 \delta\phi &= -4\pi e \left(\overline{N}_{\mathrm{i}} - n_{\mathrm{e}} + \frac{mc^2}{eB} N_{\mathrm{i}} \nabla_\perp \cdot \left(\frac{1}{B} \nabla_\perp \delta\phi \right) + \frac{mc}{e} \nabla_\perp \cdot \left(\frac{1}{B^2} J_{\|\mathrm{i}} \nabla_\perp \delta A_\| \right) \right) \\
\nabla_\perp^2 \delta A_\| &= -\frac{4\pi}{c} \left\{ \overline{J}_{\|\mathrm{i}} + j_{\|\mathrm{e}} + \frac{mc^2}{eB} J_{\|\mathrm{i}} \left[\nabla_\perp \cdot \left(\frac{1}{B} \nabla_\perp \delta\phi \right) - \frac{1}{B} \left(\widehat{\mathbf{b}} \cdot \nabla \widehat{\mathbf{b}} \right) \cdot \nabla_\perp \delta\phi \right] , \right. \\
& \left. + \frac{c}{e} \nabla_\perp \cdot \left(\frac{1}{B^2} \left(P_{\perp i} - P_{\|\mathrm{i}} \right) \nabla_\perp \delta A_\| \right) \right\}
\end{aligned}
$$

- Note the presence of higher-order moments which allude to the hierarchy problem present in gyrofluid equations.

Drift-Kinetic Limit: Energy Invariant

- The energy invariant can also be calculated in this limit:

$$E = \int d^3\mathbf{x} \frac{1}{8\pi}\left(|\nabla\delta\phi|^2 + |\mathbf{B} + \overbrace{\delta\mathbf{B}_\perp}^{\text{Perturbed magnetic field}}|^2\right) + \int d^6\mathbf{z}\frac{1}{2}m_\mathrm{e}v^2 f_\mathrm{e}$$
$$+ \int \frac{B_\parallel^*}{m} d^3\mathbf{R}dU d\mu F_\mathrm{i}(Z)\left[\frac{1}{2}mU^2 + \mu B + \underbrace{\frac{1}{2}\left(\mu B - mU^2\right)\left|\frac{\delta\mathbf{B}_\perp}{B}\right|^2}_{\text{Magnetic pumping term}} + \underbrace{\frac{1}{2}m\left|\delta\mathbf{u}_\mathrm{E}\right|^2}_{\text{Electrostatic ExB contribution}}\right]$$

- The pumping terms which keep appearing in the electromagnetic expressions are in fact due to the presence of the magnetic field in the symplectic part of the gyrocenter Lagrangian. The discrepancy between unperturbed and total magnetic field give the following expansions for the parallel kinetic energy and magnetic potential energy,

$$\mu B_\mathrm{tot} \simeq \mu B\left(1 + \frac{1}{2}\left|\frac{\delta\mathbf{B}_\perp}{B}\right|^2\right) \qquad \frac{1}{2}mU_\mathrm{tot}^2 = \frac{1}{2}mU^2\left(\frac{B}{B_\mathrm{tot}}\right)^2 \simeq \frac{1}{2}mU^2\left(1 - \left|\frac{\delta\mathbf{B}_\perp}{B}\right|^2\right)$$

Maxwellian Limit: Eikonal representation

- When dealing with Maxwellian-like distributions, it is often convenient to adopt an eikonal representation for the potential fluctuations.
- The gyro-averaging is performed in Fourier space with $\nabla \leftrightarrow i\mathbf{k}$ for the electric and magnetic potential fluctuations,

$$\delta\phi_{gc} = \sum_{\mathbf{k}} \delta\phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}} = \sum_{\mathbf{k}} \delta\phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} e^{-ik_\perp \rho_\perp \cos\varphi}$$

$$\delta A_{\|gc} = \sum_{\mathbf{k}} \delta A_{\|\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}} = \sum_{\mathbf{k}} \delta A_{\|\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} e^{-ik_\perp \rho_\perp \cos\varphi}$$

$$\langle \delta\phi_{gc} \rangle = \sum_{\mathbf{k}} \delta\phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} J_0\left(k_\perp \rho_\perp\right)$$

$$\langle \delta A_{\|gc} \rangle = \sum_{\mathbf{k}} \delta A_{\|\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \underbrace{J_0\left(k_\perp \rho_\perp\right)}_{\text{Gyro-average coefficient}}$$

- The gyro-averages reduce to Bessel functions.
- Note there is no expansion with respect to perpendicular wavenumber.

[Wang and Hahm, PoP 2010]

Maxwellian Limit: Maxwell's Equations and Energy

- We assume a Maxwellian gyrocenter distribution in the perpendicular direction.

$$F_M \propto \exp\left\{-\frac{\mu B}{T_\mathrm{i}}\right\}$$

- Taking the required moments gives the density and parallel current,

$$n_\mathrm{i} = N_\mathrm{i} - \sum_{\mathbf{k}} \exp\left(i\mathbf{k}\cdot\mathbf{x}\right)\left(1-\Gamma_0\left(b\right)\right)\frac{e}{T_\mathrm{i}}\left(N_\mathrm{i}\delta\phi_{\mathbf{k}} - \frac{J_{\|\mathrm{i}}}{c}\delta A_{\|\mathbf{k}}\right)$$

$$j_{\|\mathrm{i}} = J_{\|\mathrm{i}} - \sum_{\mathbf{k}} \exp\left(i\mathbf{k}\cdot\mathbf{x}\right)\left(1-\Gamma_0\left(b\right)\right)\frac{e}{T_\mathrm{i}}\left(J_{\|\mathrm{i}}\delta\phi_{\mathbf{k}} + \frac{1}{mc}\left(N_\mathrm{i}T_\mathrm{i} - P_{\|\mathrm{i}}\right)\delta A_{\|\mathbf{k}}\right)$$

$$\Gamma_0 \equiv e^{-b}I_0(b), \quad b \equiv \frac{k_\perp^2 T}{m\Omega^2}$$

Isotropic Maxwellian: the magnetic terms disappear.

- The corresponding global energy invariant is

$$E = \int d^3\mathbf{x}\frac{1}{8\pi}\left(\left|\nabla\delta\phi\right|^2 + B^2 + \left|\nabla_\perp\delta A_\|\right|^2\right) + \int d^6\mathbf{z}\frac{1}{2}m_\mathrm{e}v^2 f_\mathrm{e} + \int \frac{B_\|^*}{m}d^3\mathbf{R}dUd\mu F_\mathrm{i}\left(Z\right)\left[\frac{1}{2}mU^2 + \mu B\right]$$
$$+\frac{e^2}{2T_\mathrm{i}}\sum_{\mathbf{k}}\left(1-\Gamma_0\left(b\right)\right)\left[N_\mathrm{i}\left|\delta\phi_{\mathbf{k}}\right|^2 + \left(N_i - \frac{P_{\|\mathrm{i}}}{T_\mathrm{i}}\right)\left|\delta A_{\|\mathbf{k}}\right|^2\right].$$

Summary

Research Summary

- **A set of new nonlinear electromagnetic gyrokinetic Vlasov equation with polarization drift and accompanying gyrokinetic Maxwell equations was systematically derived by using the Lie-transform perturbation method in toroidal geometry. They include explicit terms existing in the drift-kinetic formalism but hard to extract from standard nonlinear gyrokinetic equations.**

- **For the first time, the drift-kinetic parallel acceleration is recovered in the long-wavelength limit from the gyrokinetic equations, validating our method.**

- **Expressions for the case of a Maxwellian distribution in the perpendicular direction to the magnetic field, with arbitrary perpendicular wavelength, have also been derived.**

Applications for this Work

- **With our new gyrokinetic formulations, the polarization nonlinearity and the usual finite Larmor radius effects related to bulk-ion Compton scattering can be treated on an equal footing to extend the validity regime further towards the long-wavelength regime.**

- **This work is instrumental for studying nonlinear interactions of intermediate mode number Toroidal Alfvén Eigenmodes which are predicted to be unstable in the kinetic regime.**

- **The gyrocenter remains closer to the particle orbit for a longer time, which is especially important in plasma regions with strong ExB shear (particularly the plasma edge, near the SOL).**

- **The model is tailored for shear-Alfvén waves (parallel magnetic potential fluctuations), but extension to full magnetic perturbations is possible without changing the overall method.**

[Duthoit, Hahm and Wang, PoP 2014]