カオス的に振る舞う磁力線の数値計算による追跡

(Numerical integration of chaotic magnetic field trajectory)

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Outline

- Background and Motivation
- Part1. Integration by Hamiltonian decomposition method
- Part2. Integration by classical method
- Conclusions

Background and Motivation

We will examine diffusion of charged particles in area with chaotic magnetic field lines.

1. Charged particles move along magnetic field lines essentially.

2. Because of finite Larmor radius of charged particle motion, diffusion of charged particles may be different from that of magnetic field lines.

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We examine difference in accuracy of chaotic magnetic field line trajectory among various numerical algorithms.

ABC flows
（Arnold-Beltrami-Childress flows)

Flow velocity is $\boldsymbol{u}(\boldsymbol{x})$,where

$$\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} u_x(\boldsymbol{x}) \\ u_y(\boldsymbol{x}) \\ u_z(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} A \sin \lambda z + C \cos \lambda y \\ B \sin \lambda x + A \cos \lambda z \\ C \sin \lambda y + B \cos \lambda x \end{pmatrix}.$$

Where $\boldsymbol{x} = (x, y, z)^T$ and λ is constant. Flow velocity $\boldsymbol{u}(\boldsymbol{x})$ can change $\boldsymbol{B}(\boldsymbol{x})$ so that magnetic field $\boldsymbol{B}(\boldsymbol{x})$ is incompressible vector field.

- A, B, C are real parameters. Flow velocity $\boldsymbol{u}$ is a periodic function of period $\frac{2\pi}{\lambda}$.
  We do not lose generality by setting λ=1

Feature

When A, B, C are not zero, magnetic field line is chaotic.

Hamiltonian mechanics and the method of Hamiltonian decomposition

Hamilton's equation

Hamiltonian is, ※

$$H[\boldsymbol{q},\boldsymbol{p}] = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{q}).$$

Canonical momentum

$$\boldsymbol{p} \coloneqq (p_1, p_2, p_3)^T,$$

Canonical coordinate

$$\boldsymbol{q} \coloneqq (q_1, q_2, q_3)^T.$$

Hamilton's equation is

$$\dot{\boldsymbol{q}} = \partial_{\boldsymbol{p}} H, \qquad \dot{\boldsymbol{p}} = -\partial_{\boldsymbol{q}} H.$$

If we define $\boldsymbol{z} \coloneqq (\boldsymbol{q},\boldsymbol{p})^{\boldsymbol{T}}$,

$$\dot{\boldsymbol{z}} = J\partial_{\boldsymbol{z}} H.$$

$$J := \begin{pmatrix} \boldsymbol{0} & \boldsymbol{1} \\ -\boldsymbol{1} & \boldsymbol{0} \end{pmatrix}$$

0,**1** are zero and unit matricies, respectively

※m is particles mass. V is potential energy.

If we write the evolution equation as

$$\dot{\boldsymbol{z}} = V_H \boldsymbol{z}.$$

The formal solution becomes

$$z(t) = e^{tV_H} z(0).$$

Hamiltonian decomposing

Let us decompose this Hamiltonian as

$$H = H_1(\boldsymbol{p}) + H_2(\boldsymbol{q}).$$

H_1 depend only on $\boldsymbol{p}$,

H_2 depend only on $\boldsymbol{q}$.

Substitution into the formal solution

The decomposed exponential operators are integrated easily and exactly.

Hamiltonian for ABC flow

Let us introduce Hamiltonian for ABC flow as

$$H(\boldsymbol{q},\boldsymbol{p}) \coloneqq \frac{1}{2}\left({p_x}^2 + {p_y}^2 + {p_z}^2\right) - AB\cos z\sin x - BC\cos x\sin y - CA\cos y\sin z\,.$$

Canonical equation gives an evolution equation giving the magnetic field line trajectory :

$$\dot{\boldsymbol{q}}(\tau) = \boldsymbol{B}\big(\boldsymbol{q}(\tau)\big).$$

Hamiltonian can be decomposed as

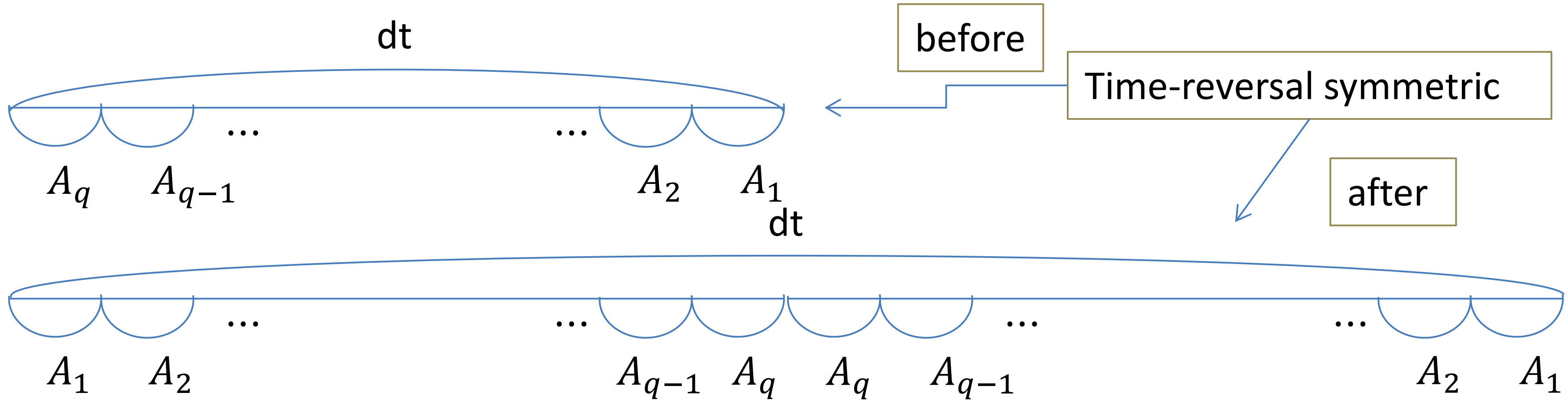
$$H = H_1(\boldsymbol{p}) + H_2(\boldsymbol{q}),$$
$$H_1(\boldsymbol{p}) \coloneqq \frac{1}{2}\left({p_x}^2 + {p_y}^2 + {p_z}^2\right),$$
$$H_2(\boldsymbol{q}) \coloneqq -AB\cos z\sin x - BC\cos x\sin y - CA\cos y\sin z\,.$$

We can apply high-accuracy numerical integration by using this decomposition. (M. Furukawa et al. (2016))

Higher-order time evolution operator

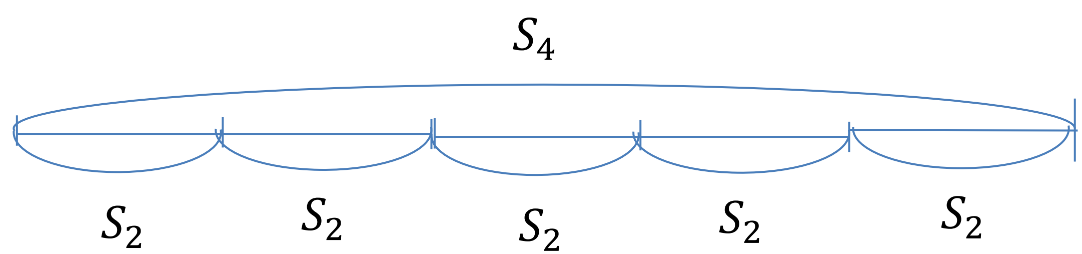
Time-reversal symmetric operator

For an exponential operator $e^{t(A_1+\cdots+A_q)}$, We obtain $e^{t(A_1+\cdots+A_q)} \simeq e^{tA_1} \cdots e^{tA_q}$ as first-order approximation. By requiring time reversal symmetry, we obtain second-order approximation:



Combination of lower-order operator gives us high-order time evolution operator

We express time-reversal 2mth order symmetric exponential operator as S_{2m}. We get high order scheme by combining lower order scheme.



大貫，鈴木，柏，“経路積分の応用”（岩波書店）(in Japanese)

Settings for numerical tests

For numerical tests of various integration algorithms, we use ABC field

$$\boldsymbol{B}(\boldsymbol{x}) = \begin{pmatrix} A\sin z + C\cos y \\ B\sin x + A\cos z \\ C\sin y + B\cos x \end{pmatrix}$$

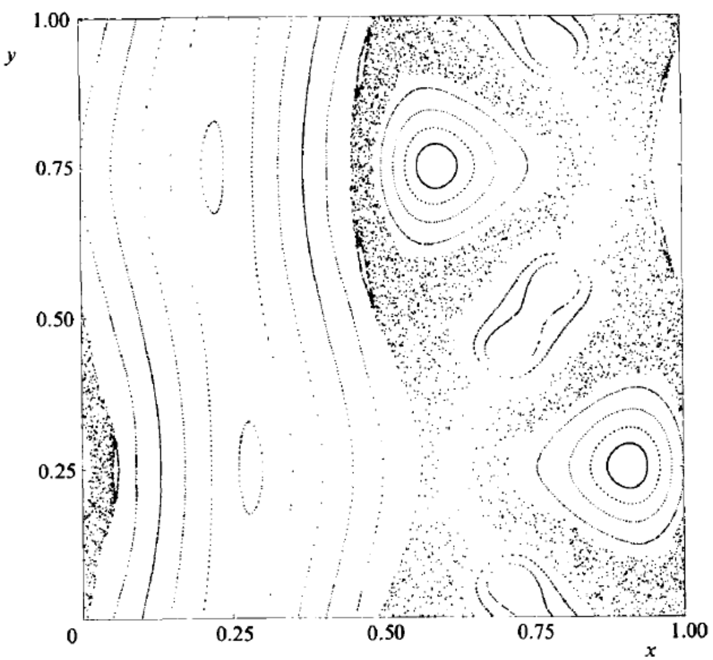
with

$$A^2 = 1, \qquad B^2 = \frac{2}{3}, \qquad C^2 = \frac{1}{3}.$$

Right figure is Poincaré-plot using

$$A^2 = 1, \qquad B^2 = \frac{2}{3}, \qquad C^2 = \frac{1}{3}.$$

(Cited from ：T.Dombre et.al.J.Fluid Mech.(1986), vol. 167,353-391)

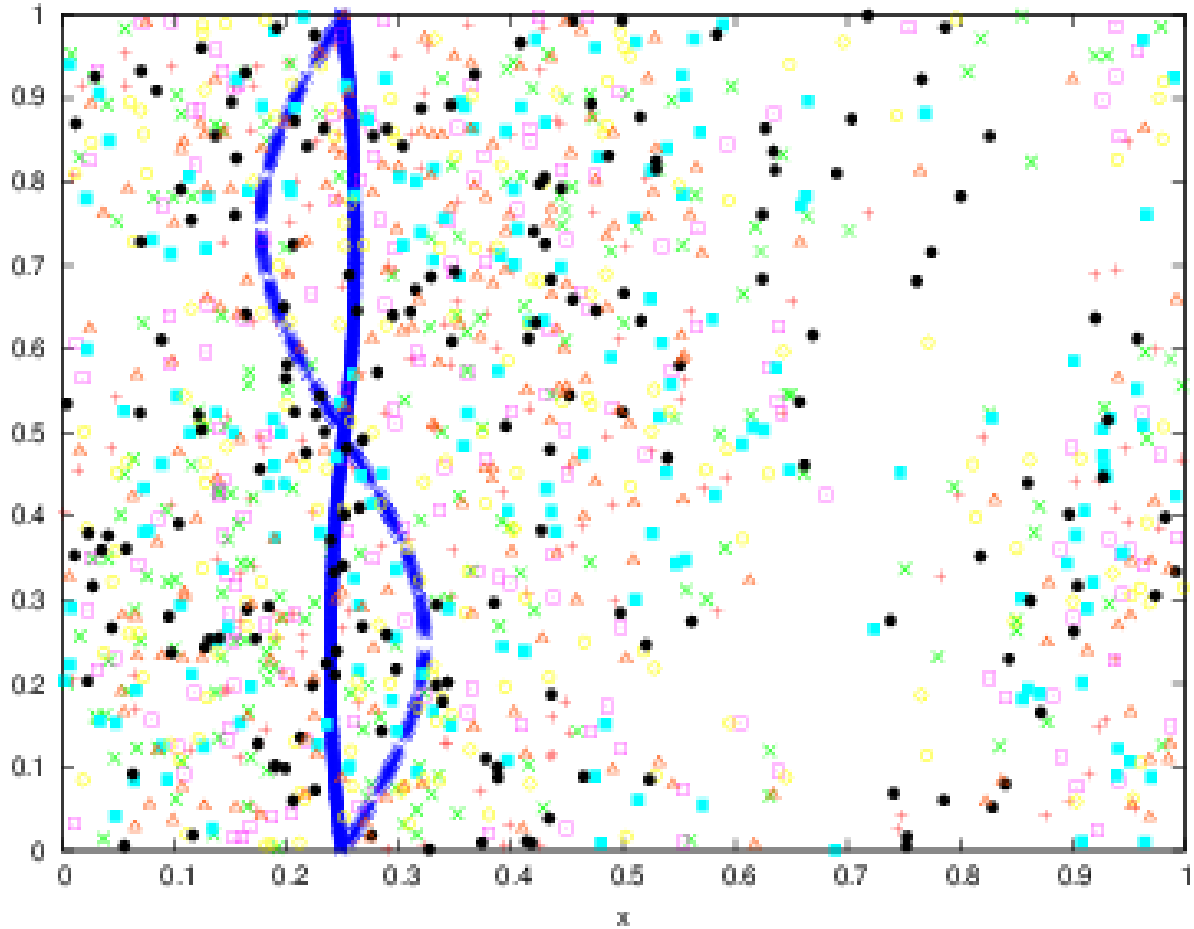


Numerical results by Hamiltonian decomposition method

Hamiltonian decomposition method was used for the Poincare plot.

We calculated magnetic field trajectory starting from $x/(2\pi)$=0, 1/8, 2/8 and so on on the x axis.

Result
We did not obtain a similar Poincare plot in the reference paper, although we observe magnetic island structure only when starting from $x/(2\pi)$=1/4.



Analysis of failure

For example, we think about $\frac{dx}{dt} = f$.

Change this equation as

$$\begin{cases} \dfrac{dx}{dt} = v \\ \dfrac{dv}{dt}\left(= \dfrac{d}{dt}\dfrac{dx}{dt}\right) = \dfrac{df}{dt} \end{cases}$$

This form is same as the canonical equation obtained by the Hamiltonian the ABC flow.

If we re-combine two equations :

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{df}{dt}$$

$$\frac{d}{dt}\left(\frac{dx}{dt} - f\right) = 0$$

Then $$\frac{dx}{dt} = f + C.$$

Therefore, we may effectively solve an equation different from that of magnetic field line by the Hamiltonian decomposition method.

Comparison among classical integration algorithms

compare magnetic field line trajectories obtained by Störmor-Verlet and Runge-Kutta methods.

Störmor-Verlet method (2nd order)

$$q_x = q_x + \frac{1}{2} dt * B_x$$

$$q_y = q_y + \frac{1}{2} dt * B_y$$

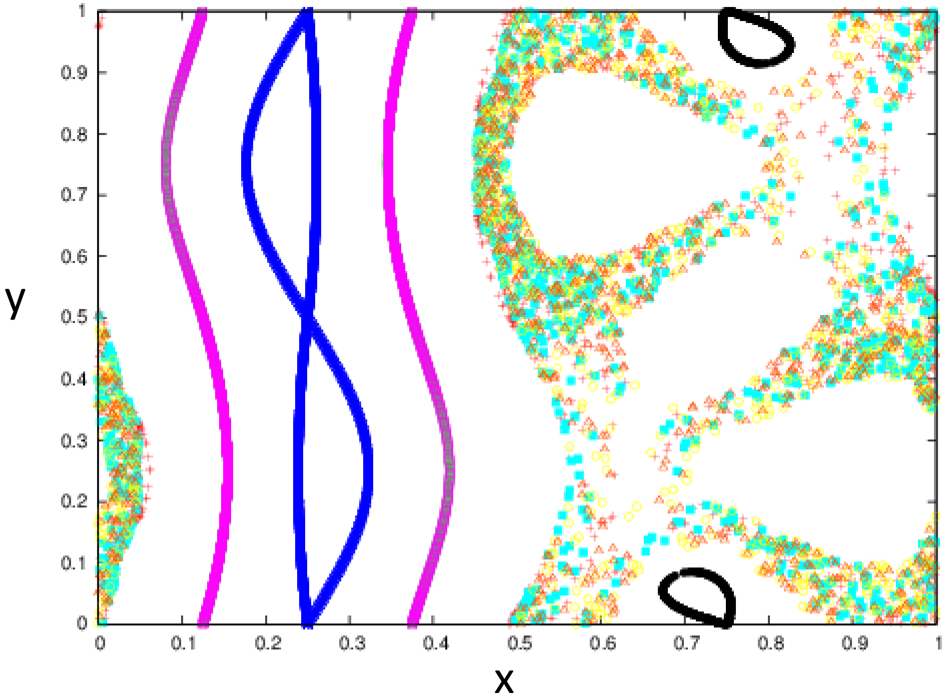
$$q_z = q_z + dt * B_z$$

$$q_y = q_y + \frac{1}{2} dt * B_y$$

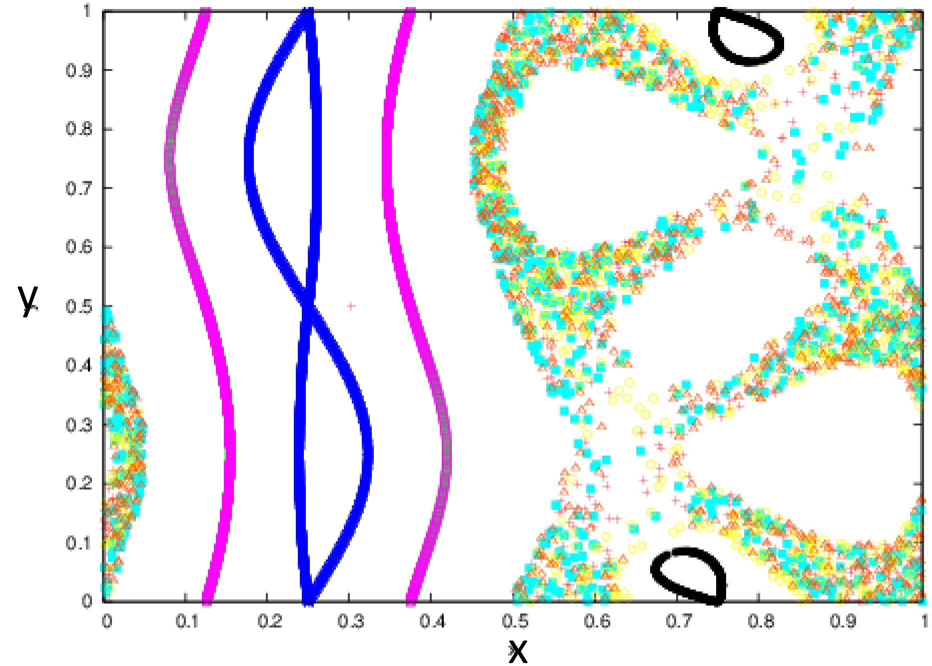
$$q_x = q_x + \frac{1}{2} dt * B_x$$

It is time reversal symmetric.

Poincare plots



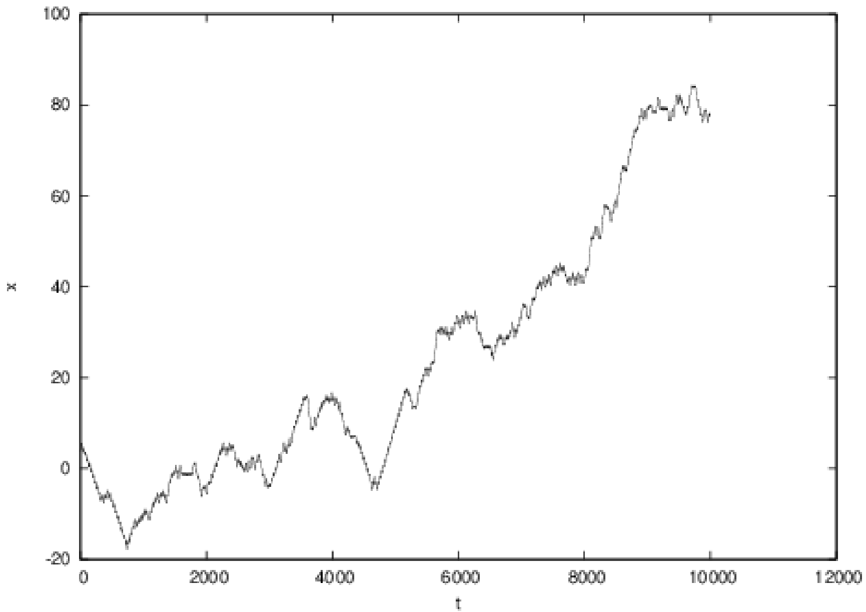
SV



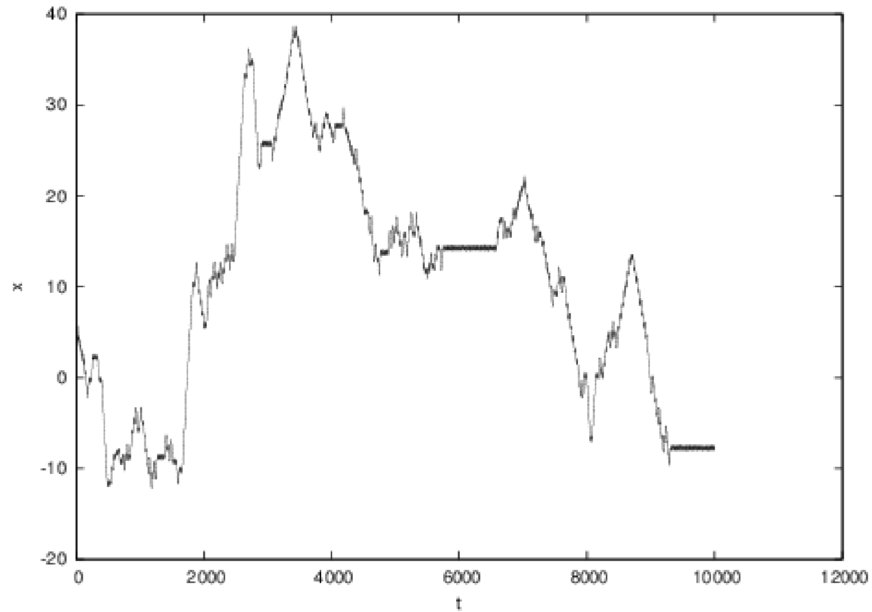
2th-order RK

We found that SV and RK are almost same as the Poincare plot in the reference paper.

Comparison



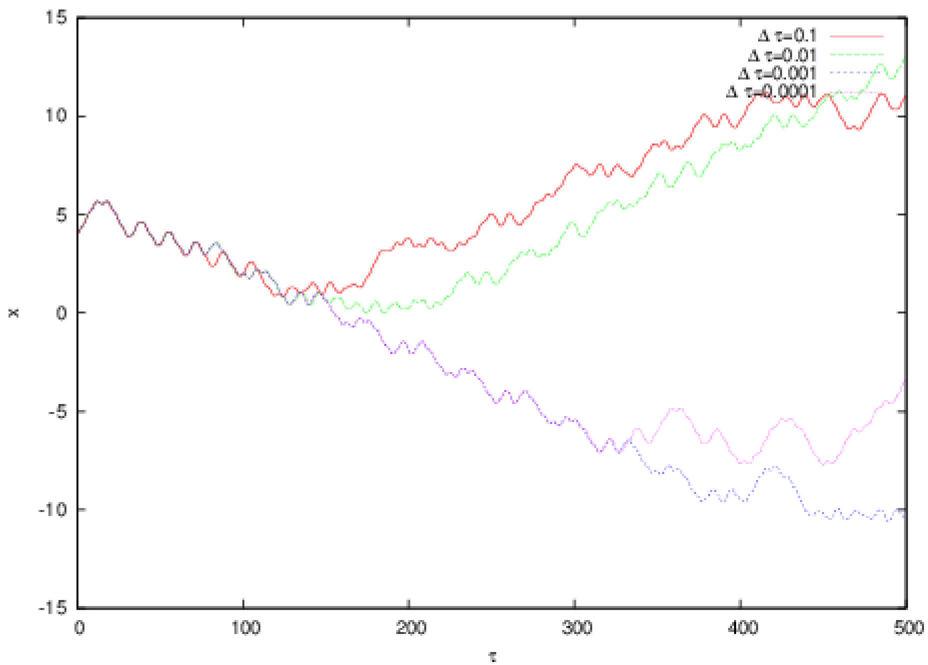
SV



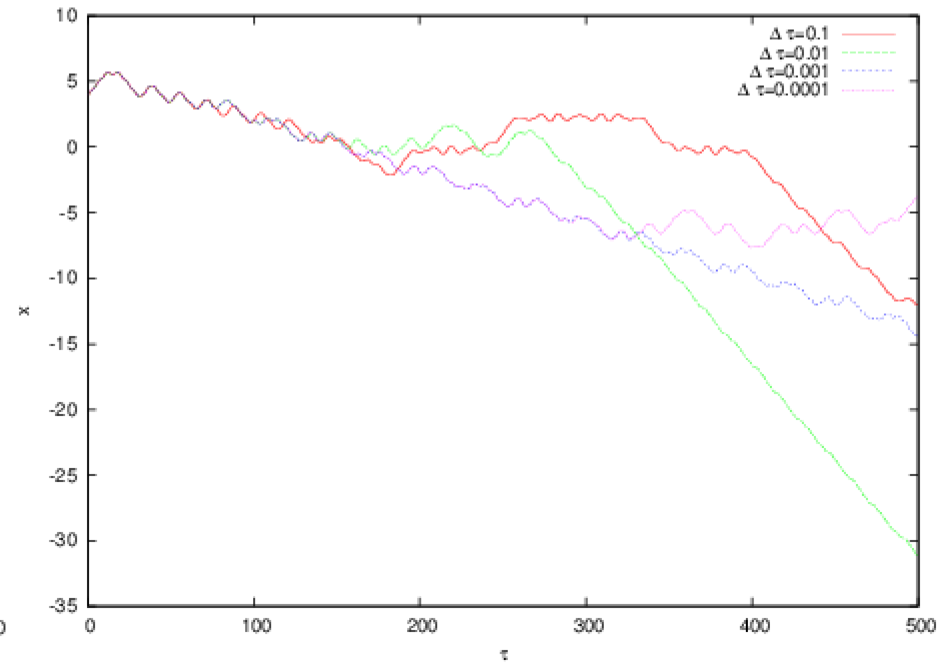
2th-order RK

(Artificial) Time evolution of x is shown for a magnetic field starting from a chaotic magnetic field region.
Two result are very different because of the chaotic nature of the magnetic field line.

Comparison when $\Delta\tau$ changed



SV



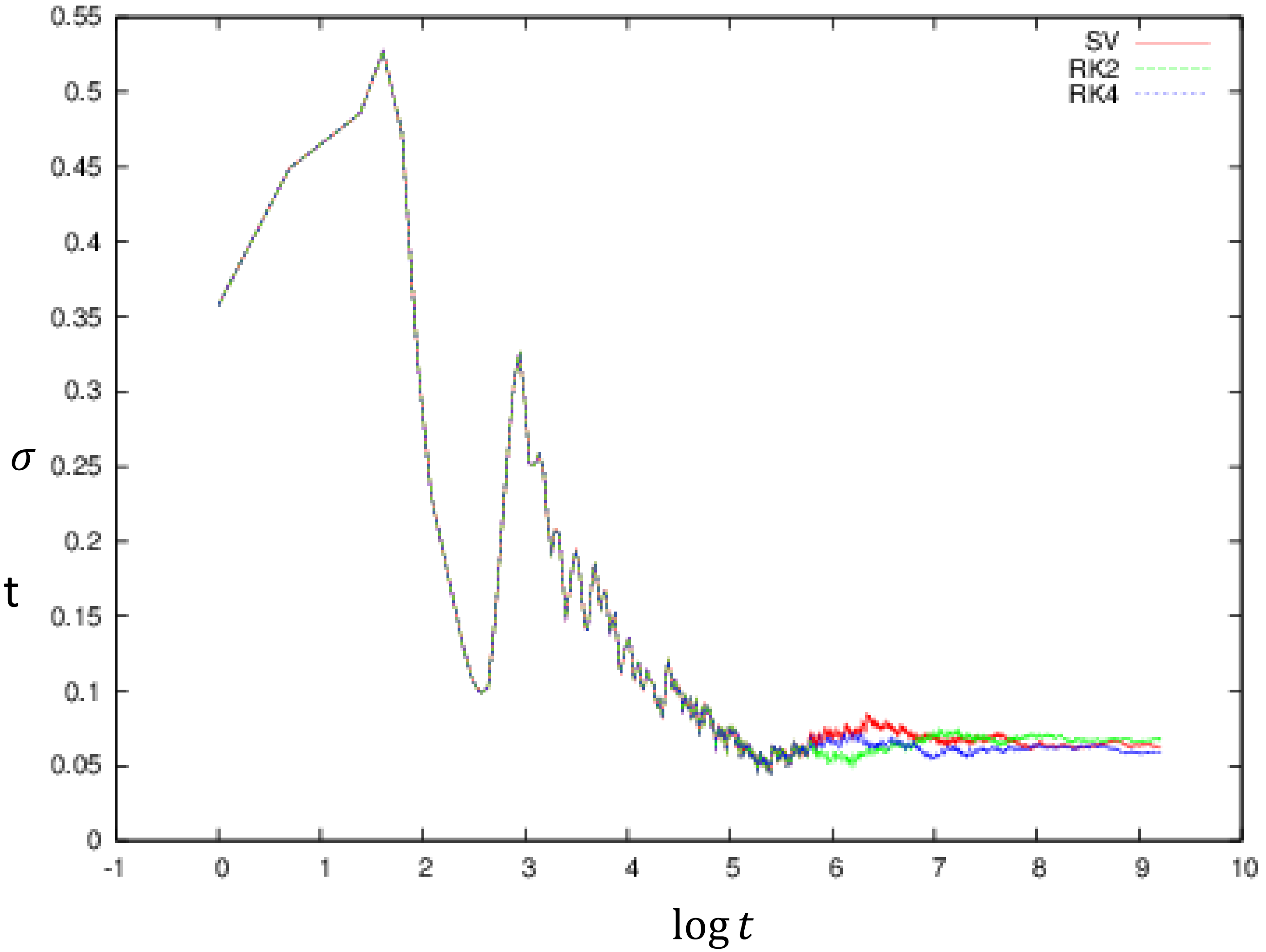
2th-order RK

The figure when $\Delta\tau$ changed . $\Delta\tau = 10^{-n} (n = 1 \sim 4)$.
The trajectories with different $\Delta\tau$ start to deviate after some time because of the chaos.

Comparison for of Lyapunov exponent

We calculated Lyapunov exponent.

The Liapunov exponent is about σ=0.07 for all algorithms adopted.



Analysis of trajectory deviation

If discretization error $c(\Delta\tau)^{\alpha}$ appears at first step , then it increases to 1 (typical scale length), after time T as

$$c(\Delta\tau)^{\alpha} e^{\sigma T} = 1.$$

Here c is a parameter depedent on the algorithm, and α is parameter about order. We solve this equation about T , we get

$$T = \frac{-1}{\sigma}(\log c + \alpha \log(\Delta\tau)).$$

For example, plugging $c = 10, \Delta\tau = 0.01, \sigma = 0.07, \alpha = 2$ in the equation,

$$T \cong 120.$$

This is good agreement with the numerical results shown above.

When $\Delta\tau$ become e^{β} times, T increse $-\frac{\alpha\beta}{\sigma}$. For example , when $\Delta\tau$ become 10 times approximately and set $\beta = 2, T$ decreases about 57.

This also reasonably agrees with the numerical results

Conclusions

- Part1
- We could not reproduce the Poincare plot in the reference paper by the Hamiltonian decomposition method. We analyzed the reason.

- Part2
- The classical method (SV, RK) succeeded to reproduce the Poincare plot of the reference paper.
- The trajectory of the magnetic field line showed chaotic behavior
- The deviation of the trajectory can be explained by the Liapunov exponent
- Since the charged particle orbit will be calculated by SV-like, time-reversal symmetric algorithm, it may be better to calculate the magnetic field line trajectory and its Liapunov exponent by SV-like algorithms